Academic Concepts as Assessed by Experience

A Memoir by Elmer G. Wiens

Introduction.

I want to examine how my experiences at an early age, especially my work involvements and travels, influenced my career path. I received university degrees in economics and mathematics with courses in computer science. As an Economist, I worked for the Canadian and B.C. Governments and lectured at universities. As an Applied Mathematician, I worked as an Operations Research Analyst in the public and private sectors and taught university courses in operations research. I also worked as a Computer Programmer / Systems Analyst at various stages of my career. My experience working with computers facilitated my university studies and academic research.

For example, while studying university economics I became aware of the concept of an economic production function, often represented mathematically as q = f(L, K, M). In this relationship captured by the function f, q is the quantity of a produced good or output, and the factor inputs, L, K and M, are the quantities of labour, capital, and materials and supplies, respectively, needed to efficiently produce that quantity of output. I reflected on my various summer jobs to see how the production processes I had encountered, could be modelled satisfactorily by a production function.

From the age of eight, I lived in the town of Yarrow, British Columbia (http://www.yarrowbc.ca/). Berry farms were the primary industry. I began picking berries at a young age at home and then for the Nightingale family at their raspberry farm next to the Vedder River. Pickers were paid five cents per pound. I learned that with “piece work” my income depended on how fast I worked.

My parents had owned a twenty acre farm a few miles outside of Yarrow, North East intersection of Browne and Vedder Mountain Roads, (https://www.egwald.ca/ubcstudent/articles/salmonstream.php). They marketed raspberries, strawberries, potatoes, corn, cow milk, beef heifers, chicken eggs, and pullets. Over fifteen years, they increased their production and income by substituting their labour with machines and hiring temporary workers. For example, they could milk twice as many cows with an electric powered Surge Bucket Milker than by hand. By adding a chicken barn and by automating manure removal, they could more than double the number of chickens and eggs laid. Labour, machinery, and buildings (capital) are substitutes in production (https://www.egwald.ca/economics/productionfunctions.php). My parents were Katie (Derksen) and George Wiens, and my siblings were Raymond, Luetta, and Alfred.

There were two high schools in Yarrow. The Mennonite Church operated the private Sharon Mennonite Collegiate with classes from grades six to twelve (http://www.yarrowbc.ca/mennoniteschools/smc.html). For grades seven to nine, I attended the public Yarrow Junior High, and for grades ten to thirteen, I attended Chilliwack Senior High. In grade eight, I was elected vice-president of the Student Council, in grade nine I was the president (http://www.yarrowbc.ca/publicschools/elementaryjuniorhigh.html). In grade ten, I was the school’s chess champion, and in grade twelve the graduating class historian (http://www.yarrowbc.ca/publicschools/pdf/chsreunion60.pdf). I was the class president and Speaker of the Student Assembly in grade thirteen. My school friends, Dave Nightingale, Rob Giesbrecht, Bob Reimer, Otto Baerg, Earl Miller, Walter Teichgrab, Mell Martens, Rita Sawatsky, Rita Paetkau, Mary Jozsa, Mary Arendt, Katie Dettling, Evelyn Paetkau, and many others continued as friends throughout my life.

After my older brother, Ray, left for UBC, I took over his Province newspaper route. I often had an extra paper which I would read before breakfast or in the evening. I read the front page and of course the comics. Articles on sports and politics interested me. Reading the daily newspaper became a lifelong habit.

There must be better ways to earn money than getting up at 5:30 am and cycling three miles to deliver papers before breakfast? Thankfully, I was offered a sales job at Eaton's department store in Chilliwack, and in the fall of 1962, I worked in Eaton's men's wear department on Friday evenings and Saturdays. This was my introduction to economics: The capitalist system, commerce, advertising, money, and credit cards. I enjoyed interacting with shoppers, and developed my skills of a sales clerk.

When I was in grade twelve, Chilliwack Senior Secondary School organized a mock parliament in March, 1963. For the election, every homeroom was a constituency contested by a candidate from each of the four political parties. The Liberals won a majority of seats, followed by Social Credit, NDP, and Progressive Conservatives. I was the Liberal campaign manager and the Minister of Trade and Commerce for the mock parliament held in the school gymnasium.

During the April, 1963 Federal Election, Lester Pearson's Liberals won the most seats, followed by John Diefenbaker's Progressive Conservatives, Social Credit, and the NDP. Chilliwack was part of the Fraser Valley constituency where Social Credit won, followed by the NDP, Liberals, and Progressive Conservatives.

On the day of the Federal election, I was a scrutineer for the Liberals at the Chilliwack Library polling station with my Social Studies teacher, Mr. Doug Steinson, and my Junior High Principal, Mr. Walter Ferguson. This wasn't my first involvement in a Canadian election. In March of 1958, I helped my mother run the Yarrow Progressive Conservatives' campaign office located in the lunch room of my Uncle Jake Derksen's berry cannery.

During the campaign the Progressive Conservative leader, John Diefenbaker, stopped in Yarrow. The townsfolk lined both sides of Central Road. Accompanied by my grandfather Julius Derksen, I stood on the curb and shook Diefenbaker's hand as he walked past us. (https://www.egwald.ca/ubcstudent/shortstories/nevil.php).

When John Diefenbaker died In August, 1979, I lived in Ottawa. I walked by his coffin as he lay in state in the House of Commons. I recalled meeting him twenty-two years before, and the speech I gave to Chilliwack Senior Secondary School's assembly during the mock election campaign. I planned to be more active in future political affairs.

In the summer of 1960, my brother, Alfred, and I picked raspberries for our Central Road neighbours, the Baerg family, at their large raspberry farm on Cherry Avenue. We would hitchhike to Cultus Lake to swim at the main beach after a hot day in the raspberry patch. Mom handed us some cash so we could go to the Pavilion and get a pop and some French fries that were dripping with salt, vinegar, and ketchup. We were always given a ride back to Yarrow by a friendly citizen. In the evenings, I occasionally served as a swamper on the truck with Mr. John Kroeker, collecting raspberries from farms in Yarrow and Abbotsford for transport to the Earl Percy and Son cannery.

Summer of 1961.

The independent strawberry and raspberry famers of my hometown cultivated enough berries to support four berry canneries.

When I was sixteen years old, I had my first job for which I was paid an hourly wage. My job at the cannery of Earl Percy and Son was to fill cans with 38 pounds of raspberries that had been dumped into a hopper from a conveyer belt.

The cannery's production process is described in detail below.

The first two paragraphs are from "Yarrow in the Summer Time" by Elmer G. Wiens: http://www.yarrowbc.ca/settlers/settlers1956_65.html#yarrowsummertime

"During the strawberry and raspberry harvests, Yarrow's canneries hired local adults and older teenagers to process the berries. Growers and truckers transported flats of berries to the canneries' loading docks. At the cannery, each farmer's berries were weighed on a large scale, inspected for their quality, and the farmer was given a receipt for the net weight of berries delivered. The stacks of berry flats were set-aside on the spacious loading dock for the night's processing. A typical cannery had two conveyer belts used to sort-out the bad berries. A muscular teenager dumped flats of berries onto the lower end of a revolving belt. Overhead nozzles sprayed and cleaned the berries as they moved up the conveyer belt. Three women standing or sitting at each side of the belt removed rotten and green berries, twigs, leaves, dirt, and insects. At the top end of the belt, the berries entered a large hopper—a tin container tapering downward. A worker controlled the flow of berries from the hopper into a can with a sliding metal plate. Another worker weighed the filled cans to ensure each can contained 38 pounds of berries. Then the cans moved along another conveyer to a waiting truck for transport to a cold storage facility. Poor quality berries were emptied into barrels containing four hundred pounds, which were preserved with a SO2 solution, sealed, and stored for delivery to the British overseas jam market."

"At the height of the raspberry season, cannery workers had long shifts until all of the day's berries were processed. These shifts often started at five o'clock in the afternoon, and continued until eight or nine the next morning. Young men assigned to the clean-up crew would then work another hour, washing down the machinery and conveyer belts, sweeping the floors and loading docks, and cleaning the washrooms. Tired cannery employees had just enough time to go home for a meal and sleep, before returning to the cannery for another shift."

Let's examine the operation of a raspberry cannery in more detail.

The entrepreneur was the owner of the company that operated the berry cannery, including the capital, and the materials and supplies used for processing the berries. The entrepreneur had an agreement with farmers regarding acceptance of their delivered supply of berries, to be paid for at the end of the season. He also entered into an agreement with companies willing to purchase a quota of canned berries (e.g. Birds Eye in the USA, founded by Clarence Birdseye – the 1920 inventor of the quick-freezing method for frozen food).

Let's break down the berry cannery into its Output and factors of production: Capital, Materials and Supplies, and Labour, other than the entrepreneur.

Output: Cans of clean berries weighing 38 pounds each. Loaded onto trucks and transported to a buyer's cold storage facility.

Capital:

1. The Berry Cannery Building was a converted Lumber Yard Sales Building, with a receiving loading dock and a shipping dock with a bay for trucks, that contained an office, lunch room, and wash rooms.
2. The land -- the former lumber yard, about two acres.
3. Two conveyer belts with overhead water nozzles, and electric motors to rotate the belts.
4. Two metal shakers that spread berries dumped from flats evenly onto the conveyer belts.
5. Two trucks for hauling cans of berries to cold storage and for picking up berries from farmers.
6. A short conveyer belt that moved filled 38-pound cans to the deck of the delivery truck.
7. Two weigh scales, one for the framers' stacks of flats filled with berries, the other to weigh the cans of processed berries.
8. Three L-shaped hand trucks with a low flat platform and two large wheels.
9. Office equipment like filing cabinets, telephones, desks, typewriters, etc.

Materials and Supplies:

1. Enough wooden flats with 12-pint baskets (hallocks) provided to the farmers for at least 2-3-days processing at the height of the berry season.
2. Enough cans for delivering processed berries to cold storage for the entire berry season.
3. A season's supply of berries as contracted with and provided by the local berry farmers.
4. A plentiful supply of clean water.
5. Electricity to power the building and loading docks, and run the electric motors.
6. Receipt booklets to record the weight of berries supplied by each farmer per day---copy to farmer and cannery.
7. Office supplies like stationery for the business's day-to-day operations.

Labour:

1. Loading deck: one man to weigh the flats of berries; grade the berries as to quality, and issue receipts.
2. Loading deck: older teenager to assist farmers to load flats of berries from pickup trucks and/or trailers onto the loading deck, move stacks of flats onto the scale, and supply farmers with enough empty flats for the next day's picking.
3. Loading deck: two older teenagers who used the hand trucks to move stacks of flats from the scale to temporary storage; to move flats of berries to the dumpers, and to remove stacks of empty flats.
4. Twelve women (cleaners): Six women per revolving conveyer belt to ensure quality of berries and to remove rotten and green berries, twigs, leaves, dirt, and insects.
5. Two men (dumpers) standing at the head of the conveyer belt dumped flats of berries onto a metal shaker that deposited the berries onto the revolving conveyer belt.
6. Two teenage boys (canners) filled cans with 38 pounds of berries from the hopper being filled by each conveyer belt.
7. One older teenager (weigher) to weigh the cans to assure they contained 38 pounds of berries.

8. Two truck drivers and two strong helpers (swampers) who assisted the truck drivers by loading and unloading the trucks.
9. One man: the cannery foreman who supervised the entire process.
10. One woman: the supervisor of the conveyer belt workers. She had the authority to stop or slow down the belt to ensure the berries were washed and clean of foreign debris.

Valuing the output and factor inputs:

1. Output: Agreed contract price per 38-pound can containing processed berries adjusted as to the quality of the delivered berries.
2. Capital: Replacement cost depreciated as to the present condition of the factor. Confirmed by the municipalities assessed value of the capital items of the business.
3. Equipment and Supplies: Cost of purchase.
4. Labour: Hourly wage rate times number of hours worked per day. The wage rate depended on the worker's job.

Historical Note: When the cannery was processing berries for delivery to the Birds Eye Company, a representative of the company was on hand at the cannery to ensure the quality of the berries to be delivered.

The cannery's operating parameters gave the foreman the option of running one or two conveyer belts to process the day's supply of berries depending on the quantity of berries received from the farmers. Either way, the cannery would operate until all the delivered berries were processed. Shifts could vary from four to fifteen hours per night, usually starting at 5:00 pm.

When running one conveyer belt, six cleaners, one dumper, one hand truck operator, and one canner were not needed. However, the other costs per hour of the cannery's operation, and thereby these costs per pound of berries delivered, were still incurred. Therefore, running two belts resulted in lower costs per pound of berries delivered.

When berries were limited in supply at the start or end of the raspberry season, the cannery would run one conveyer belt for about four days. It would run two belts for two or three weeks at the height of the season.

The cannery's production process was basically an "assembly line," with a sequence of workers carrying out designated tasks as the raspberries moved from the loading deck to the conveyer belt, ending up as processed cans of berries on a truck carted to the cold storage facility.

An assembly line processes can be modeled by the Leontief production function (https://en.wikipedia.org/wiki/Leontief_production_function), "characterized by fixed proportions of inputs with no substitutability between them."

Each worker (input) in the sequence of "work stations" at the cannery is essential to the processing of the berries. For example, one cannot substitute an hour of a dumper's time for an hour of a canner's time, or vice versa.

Going from one conveyer belt to two conveyer belts with the additional workers was necessary to increase the output of berries over a given time period.

During my personal experience, some of the workers involved were: The entrepreneur Mr. Jake Derksen, loading dock weigher Ed Derksen, dumpers Ray Wiens and Rudy Baerg, supervisor Mrs. Mary Froese, cleaner Luetta Wiens, truck driver Mr. Kroeker, can weigher Harold Froese, canners Otto Baerg and Elmer Wiens.

Summers of 1962, 63, 64.

During the next three summers, I worked for Clearbrook Frozen Foods at their Glacier Cold Storage plant, first two summers with Bob Reimer and Dave Nightingale, last summer with Mel Martens. Our boss was the congenial Mr. Rudy Reimer who skilfully managed his inexperienced crew of teenaged Clearbrook girls and Yarrow boys.

The scale of the canneries in Yarrow allowed the cannery managers to rely on local, independent farmers for their supply of berries. Clearbrook Frozen Foods' large berry and vegetable processing plant in Clearbrook, British Columbia, required a larger, and more secure source of inputs, along with inputs provided by local farmers. The company cultivated large acreages of berries and vegetables, like beans, for this supply.

At the main cannery facility in Clearbrook, berries were processed in a manner similar to the process described at the Yarrow cannery. After this, the berries were processed further. This plant produced fresh berries in boxes to be sold frozen in supermarkets. The automated production process dropped berries into 14-ounce leak proof containers, and covered them with a cold syrup of sugar and water, leaving some space at the top. The containers were sealed and placed into metal trays.

Stacks of trays on pallets were trucked to the Glacier Cold Storage plant. The truck driver or swamper moved these pallets of trays off the truck to the plant floor with a manual pallet truck (jigger). A plant worker pushed the stacks of trays with an electric or manual pallet truck into Glacier's "Quick Freeze" tunnels for flash freezing. Once frozen and pulled from the tunnels, the trays of cans were placed on tables by teenage boys. Teenage girls placed the cans into the shipping boxes. Another older teenager loaded the filled boxes onto wooden pallets three high, and with an electric pallet truck stored them in the gymnasium-sized, cold storage room. Later, a fork truck was used to stack these pallets of boxes up to four loaded pallets high.

A fork truck driver needs good three-dimensional perception (depth, width, and height), much like a basketball player making three-point jump shots over the up-stretched hands and arms of their defenders.

I drove the fork truck in summer of 1964. Four years later during the 1968-69 academic year, I taught Calculus and Analytic Geometry to 3rd year university studies while completing my M.Sc. in Mathematics. My practiced spatial awareness allowed me to visualize such three-dimensional concepts as the triple integral of three-dimensional mathematical objects.

The production process that took trays of containers from the trucks to their cold storage in boxes was accomplished by people allocated to a sequence of specific tasks. These tasks were done in a specific order and with a specific number of workers. This sequence of tasks was unlike an assembly line in a spatial sense with a fixed number of workers assigned to a rotating conveyer belt.

This process was more flexible. If four workers could box a certain number of cans in an hour, eight workers with twice as many tables could box twice as many cans in the same time period.

The “assembly line” process at the Cold Storage plant is again modeled by the Leontief production function. The ratio of workers time, boxes, and tables, the inputs, was in fixed proportion to the number of boxes of cans transferred to cold storage, the output.

Furthermore, Clearbrook Frozen Foods was an integrated company from their farms all the way to finished products sold to grocery stores and supermarket chains. Growing their own berries and vegetables had the advantage of securing a stable supply chain instead of relying on independent farmers. Moreover, the economies of scale from the large acreages cultivated by the company made input costs lower than buying these inputs from independent farmers cultivating much smaller acreages.

Many years later as a visiting Assistant Professor at Carleton University in Ottawa, I may have subconsciously used some of these experiential insights in my 1978 theoretical paper, “Government Firm Regulation of a Vertically Integrated Industry,” https://www.egwald.ca/wiens/elmerwiensgovregvertintindustry.pdf. I used this theoretical analysis to investigate a role for Petro-Canada in my 1979 paper, “Petro-Canada, Antitrust Legislation, and Vertical Integration in the Canadian Petroleum Industry,” Discussion Paper #41, Center for the Study of Organization Behavior, University of Pennsylvania, https://www.egwald.ca/wiens/elmerwienspetrocanada.pdf.

Salada Foods bought Clearbrook Frozen Foods’ production facilities in 1964. In 1970, the large supermarket chain, Safeway Canda, bought these production facilities, integrating upstream to secure a supply of frozen and canned fruits and vegetables.

Summer of 1965.

I was a second-year student at the University of British Columbia during the 1964-65 academic year. I shared a basement apartment at 2746 West 37th Avenue in Kerrisdale, Vancouver with Vic Fast and Alvin Hamm, and Rob Giesbrecht when Vic moved out. With downtime in the afternoons, I would sometimes encounter Ed Dahl, a friend from Yarrow, while working out in the weight room of Thunderbird Stadium. He shared some experiences from the previous summer while he worked as a casual Canada Customs Officer at the Huntingdon-Sumas border crossing. That spring, he told me that representatives of the Canadian Government were interviewing for potential employees and suggested I apply for a summer job with Canada Customs.

This seemed like a more intellectually challenging job than driving fork truck for Clearbrook Frozen Foods for another summer, so I filled in an application form at the UBC Student Center. During my interview with the Customs officials, I was asked a number of questions regarding my permanent place of residence and some questions on current affairs. Thankfully, I was offered a job as a Customs Officer for the next summer. I had the option of working at the Alcan-Beaver Creek Border Crossing along the Alaska Highway in the Yukon Territory, or at the Sumas-Huntingdon Border Crossing near Abbotsford, BC.

Working in the Yukon Territory was untenable for a number of reasons. For a university student from the Fraser Valley, it was far away from home. I did not yet own my own car. The Beaver Creek Customs and Immigration Port was located in the center of the community, making control of passing traffic to the USA difficult. Most importantly, my single parent mother was in the hospital. It was important for me to spend the summer at home in Yarrow, so that my younger siblings who were still attending high school could live with me until our mother returned home.

Naturally, I chose to work at the Sumas-Huntingdon Border Crossing. My first order of business was to buy a car, a 1953 Chevrolet 210 2-Door Sedan.

On my first day on the job in May of 1965, I gave my oath for the faithful performance of the duties of a Customs and Immigration Officer, and I was issued a badge and uniform complete with two blue shirts and a tie. I was informed that I was assigned to the Lyden-Aldergrove Port in Langley, B.C. Regrettably, this border port was about 40 kilometres from Yarrow, about twice the distance to the Sumas-Huntingdon Port. But since the Port was only open from 8:00 am to midnight each day, I would not have to work the graveyard shift, like the other new recruits at the Sumas-Huntingdon Port.

The Aldergrove port of entry consisted of a squat two-story building. On the east side were two lanes for cars entering Canada northbound, on the west side, two lanes for cars leaving Canada southbound. The inner lane on each side was covered by a roof. The outer lanes were not covered so that buses, trucks, camper vans etc. could pass due to their height.

One large room facing south on the ground floor of the building had counters on three sides. The enclosed inner area was for officers; the outer area for travellers. The entire south wall had windows facing the USA Port of Entry building, so that officers could monitor traffic to and from Canada.

In 1965 Canada Customs and Canada Immigrations were separate agencies. The office of the Canada Customs administrator was on the ground floor in the north-west corner. The office of the Canada Immigrations administrator was in the north-east corner. All other officers working at the port were both Customs and Immigration officers.

Prior to the January 1965 Auto Pact between Canada and the USA, there had been a 35% tariff on all cars imported into Canada from the USA. Cars were 50% more expensive for Canadians than for Americans. Therefore, there was a monetary incentive for people to smuggle cars, new and used, into Canada.

To contain this problem, a Customs Officer would record on a Port of Aldergrove receipt the date of entry, and the make, model, year, and licence plate number of each vehicle, with USA State plates entering Canada. One copy was given to the driver of the vehicle, the other was retained and filed at the port. The driver was instructed to hand in the receipt at the Canadian border port when leaving the country. These regulations were still enforced in the summer of 1965.

After orientation to my duties as a Canada Customs and Immigration Officer (Casual), my first task was to collect the vehicle receipts from Americans exiting Canada. After I stamped the receipts with "Port of Aldergrove" and the date of the vehicle's exit, I sorted them according to the border Port where the vehicle had entered Canada. We regularly sent bundles of these vehicle receipts to the registered Port of entry and received bundles of Port of Aldergrove vehicle receipts collected at various Ports across Canada where vehicles exited.

The Ports' vehicle receipts were imprinted in an alphanumerical sequence. My next task was to pair the entry vehicle receipts on file with the exit vehicle receipts. My job was to attach receipts that matched and place them in a separate file. Unmatched vehicle receipts remaining on file represented vehicles that were still in Canada. A Canada Customs agent or an RCMP officer could use the licence plate number on the receipt to track down any American owners of vehicles remaining or sold illegally in Canada.

Collating vehicle receipts was a straightforward task, but what was the best way to compare them to those on file? During the late spring and summer months, hundreds of American vehicles entered Canada at the Port of Aldergrove every day. We received hundreds of vehicle receipts from other Ports imprinted with "Port of Aldergrove," along with vehicle receipts from Americans who had both entered and exited our port.

The filed receipts were in alphanumerical sequence. I started with one stack of receipts from one Port, and matched each receipt with those on file. This was time consuming. For each receipt I had to search through the file for a match. A better way was to sort a stack of receipts into sequence, and then match them with the filed receipts, requiring just one pass through the file per stack. This took less time.

Was there a better way? After a few weeks I settled on the following method. I sorted each stack of receipts into sequence. I merged pairs of the sorted stacks into larger stacks. These stacks were again merged in pairs. I repeated this process until one large stack of receipts in sequence remained. Then, just one pass through the file was required to find a match for every receipt in that stack.

When I took Computer Science courses during the 1965-66 academic year at UBC, I learned this was a recursive sort and merge algorithm commonly used to sort items with computers.

Ensuring that the regulations of Canada Customs and Canada Immigration were being met, officers also provided tourist services to people entering Canada. During late May and early June, I was given another job. Americans could park their cars and enter the Port building. I would hand out maps of B.C. and sell fishing and hunting licences over a counter. Our licences were priced in Canadian money. The tourists only had American money. I would use the exchange rate of US $0.9250 to compute the Canadian value of the denomination of the American money offered as payment. Then I would payout the difference in Canadian currency. We got American money. They got Canadian money and coins.

Firearms, particularly hand guns, are prohibited from entering Canada with a few exceptions. Americans holding a valid B.C. hunting licence could bring in their hunting rifles during the hunting season if properly stored in their vehicle. Americans enroute to Alaska could transport firearms, but hand guns needed special attention. The traveller was issued a copy of a receipt describing the gun which was then put into a sturdy transparent bag and sealed. When exiting Canada, the traveller was required to report in to the Port to allow a Customs Officer to match the gun with the receipt's description, and remove the seal from the bag. The hand gun receipts were managed like the vehicle receipts to ensure that guns did not remain in Canada.

Depending on the length of stay in the USA, there was a limit to the value of goods that Canadians could bring back duty and tax free. Travellers exceeding the limit had to pay a tariff, depending on the item. Only a specified quantity of alcohol and/or tobacco products were allowed returning Canadians after a forty-eight hour stay in the USA. This limit was strictly enforced.

A Canadian over their duty-free limit was required to enter the Port building with their purchased goods and receipts. An officer would consult the tariff binder for each item to determine the duty owing. I trained for this job. It was surprisingly complicated to establish the tariff categories for various household goods.

Traffic across the Canada-USA border increases significantly during the months of July and August. The permanent Port officers, like all employees, prefer to take their holidays during summer months. I was hired to fill in for these vacationing officers, and to assist with the surge in travellers crossing the border.

The Port of Aldergrove handles both passenger and commercial traffic. Trucks loaded with industrial and commercial goods were shunted to a nearby location, where they were processed by a Customs Officer specializing in these regulations and tariffs. A significant portion of them were not final products sold by wholesale or retail stores. They were intermediate goods used in making other products. For example, large loads of hay on semi-trucks were delivered to B.C. farmers to feed their livestock.

Canadians buying goods in the USA, like companies importing American products, result in a demand for American money and a supply of Canadian money. Americans shopping in Canada are like Canadian exports to the USA, and result in a demand for Canadian money and a supply of American money.

The supply and demand for a country's money is reconciled on the foreign exchange market. With a floating exchange rate, the value of the Canadian dollar adjusts with respect to the value of another country's money to equate demand and supply. With a fixed exchange rate, the Canadian government buys or sells Canadian dollars in exchange for the other country's money to make up the difference.

Canadas has a balance of trade surplus when the value of exports exceeds the value of imports. It has a balance of trade deficit when the value of imports exceeds the value of exports. Canada's current account reflects its net income from exports and imports. Its capital account reflects the net value of investments in Canadian assets by foreigners and investments by Canadians in foreign countries' assets.

One memorable event that summer was hiking into Liumchen Lake with my fellow customs agents. I had told them of a lakeside cabin where Yarrow friends and I had stayed during a previous summer. The lake was near the USA border in the Liumchen Ecological Reserve. They wanted to checkout the trails from the USA and fish for trout.

As a UBC graduate student in economics in the 1973-1974 academic year, I took courses in international trade and finance. Until that time international trade theory focused on trade in final products like new cars and groceries. My experience observing the flow of intermediate products motivated me to write the paper, "A Survey of Intermediate Products in International Trade," (https://www.egwald.ca/internationaleconomics/pdf/intermediate_products_international_trade.pdf195) for the international trade theory course.

After the 1965 Auto Pact cancelled tariffs on cars and parts, Canadian companies increased the production and export of cars. Canadian cars became less expensive to manufacture partly because parts, intermediate goods, were duty free.

During my first two months at the Port of Aldergrove I performed clerical jobs, freeing time for the permanent officers to handle vital customs and immigration tasks. Toward the end of June, senior officers instructed me about an officer's responsibilities when interviewing travellers who want to enter Canada.

Residents of Canada are classified as Canadian citizens, permanent residents, or temporary residents. An officer can ask for documents to prove legal status. Furthermore, an officer must determine whether the traveller meets the duty-free, alcohol and tobacco limits.

For others, an officer must decide if a visitor's stated purpose is proper and funds are sufficient to be admissible. Permanent residents and citizens of the USA can enter without a visa, although they could be asked for proof of status. Visitors from other countries need a passport and possibly a Temporary Resident Visa (TRV) obtained at a Canadian Consulate.

A person can apply for permanent resident (immigrant) status at a Canadian Port of Entry. A permanent resident or citizen of the USA, who is inadmissible for some reason, can submit a Temporary Resident Permit (TRP) application at a Canadian Port or Consulate. Inadmissible people from all other countries must apply at a Canadian Consulate for a TRP. The Port Administrator for Canada Immigration handled these applications.

People usually arrive at the Port of Aldergrove in vehicles. At the start of July, two lanes were open to handle the influx of visitors and returning Canadians. For two weeks I accompanied an officer on the road to enforce the customs and immigration regulations. We had to establish the admissibility status of the driver and passengers of every vehicle, including buses.

Generally, Canadian residents meeting the customs regulations enter Canada without a problem. We issued a vehicle permit to a driver of a car with USA state licence plates with instructions to hand it in at the Port of exit. Visitors were told to take their belongs with them when leaving Canada.

We were especially vigilant watching for contraband drugs, firearms, alcohol and tobacco. We would examine the trunk of a car if it seemed the returning Canadian had been on a shopping spree in the USA. Recreation vehicles, like pickup truck campers, motorhomes, travel trailers, and camper vans, might receive extra scrutiny for prohibited goods. All bus passengers could be subjected to detailed questioning about nationality, length of stay, suitcase contents, and personal items.

During the evening shift when traffic was light, I eased into "working the road" as we called the activities of an officer interviewing travellers. An officer interacts with a broad spectrum, both Canadians and Americans. Their attitudes differ somewhat because Canadians have the right to return to Canada, while it is a privilege for Americans to visit. Most people were not a problem, because they were knowledgeable about Canada's customs and immigration regulations.

At the start of a shift, I checked the file for individuals and vehicles to watch out for based on recent reports from Canadian Ports, and Canadian and American police forces. The file contained descriptions and photographs of people, and the licence plate numbers and make of vehicles, along with explanations for their unwelcome status.

A widely promoted rock concert was held in the City of Aldergrove. As the date of the concert neared, Ports in BC received reports from USA State Police Forces that San Francisco bikers, Hells Angels, were on the way to attend. The American police monitored their progress toward Canada. Officers at the Port of Aldergrove were on alert for their arrival.

Canada Immigration deemed them inadmissible. Port officers were instructed to refuse them entry to Canada. I was working the road when a group of five men on motorcycles wearing biker regalia arrived. After asking the usual questions, I determined they had insufficient funds for their declared stay and instructed them to return to the USA. Another officer and I watched as they presented themselves to the USA Port Officer.

That summer the union representing workers at BC's major breweries went on strike. Liquor stores, bars, and pubs ran out of beer. Only Ben Ginter's Tartan Brewing Company in Prince George was operating. Thirsty BC residents had the option of Uncle Ben's beer or bringing beer back from the USA. Port officers checked car trunks more frequently and were often surprised by the identity of those attempting to smuggle beer.

Fall 1965 to April 1967.

During UBC's 1965-66 academic year, I was a third-year student and lived at Room 16, Hut 10 at the Fort Camp residences. My single room, possibly the smallest on campus, had a desk, book shelf, chair, open close rack, and a dresser with two drawers under my bed. It was my first bedroom that I did not have to share with brothers or roommates, a room of my own. Known collectively as Dog-Patch, Hut 10 and Hut 11 were connected by an common area consisting of two shower stalls, a bath tub, half a dozen sinks, and one toilet. My friends from UBC's campus residences were Alec Nazaruk, Bob Dugas, George Galbraith, Rick Waller, and Alvin Hamm from Dog-Patch, and Lynn Ehrstein and Inga Fredrickson from Mary Bollert Hall.

On the advice of my older brother, I enrolled in computer science courses along with my honours mathematics courses. He had a B.Sc. in mathematics and worked as a computer programmer for Pacific Tabulating and Statistics (PTS) in Vancouver. I diligently studied the stack of manuals he gave me on programming their Univac 1050 computer in the PAL assembly language. These manuals introduced me to Regent, a higher-level report generating language that could be used with PAL.

When my brother went on to work for the Chevron Oil Company, he suggested I apply for a summer job as a programmer at PTS. I applied, and after writing a test program in PAL I started work in May. To write programs for the UNIVAC 1050 computer, I had to understand its architecture and how PAL was translated into its alphanumeric machine language.

The UNIVAC 1050 computer's hardware consisted of two arithmetic registers, a multiplier and divider, and several index registers for accessing memory locations. Input on punched cars passed into the computer through input channels. Output passed through output channels and printed onto computer paper or punched onto cards. A computer operator enabled these procedures with UNIVAC's control panel.

Keypunch operators punched the PAL program onto cards. After reading these cards, the computer assembled the PAL coding into a 30-bit machine language it punched onto color coded cards. These cards were placed in front of data cards and fed into the computer. It processed the data with the internalized program to produce reports on the printer.

The information on the data cards came from the clients of PTS. Our Accountant / Sales Representatives devised ways to translate this information into data for the keypunch operators.

PTS was one of the first companies in Vancouver, BC to use computers. Its first computer was a commercial solid-state drum computer, the venerable UNIVAC SS 80, still in use in 1966 on legacy applications. Its offices and computer facilities were located in the basement and second floor of the Marine Building. In September, the reorganized company moved to the old Province newspaper building and renamed the National Data Center.

During my fourth year at UBC, I worked as a programmer at this venue every Wednesday afternoon. On the other days, I pursued my studies in mathematics, statistics, computer science, and numerical analysis. I learned how scientific problems, like finding the solution to a system of ordinary differential equations, can be solved with the Runge-Kutta method using the Fortran programming language.

Many years later during the 2000s, I used the Runge-Kutta method to obtain solution trajectories for complex phenomena in nonlinear dynamics (https://www.egwald.ca/nonlineardynamics/index.php). For example, the Kaldor Business Cycle Model investigates "the interaction between investment and savings to produce cyclical movements in income and capital" in a hypothetical economy with a nonlinear dynamic model. (https://www.egwald.ca/nonlineardynamics/limitcycles.php#kaldormodel).

Taking courses at UBC and programming at PTS were complementary activities.

While at UBC I learned how the Critical Path Method (CPM) can be used to ensure projects are completed on time and budget. A project is broken down into sequences of tasks that can be done in parallel. CPM can identify the sequence that takes the longest time, called its critical path. A project manager can monitor the critical path's tasks to guarantee its efficient completion.

With my own initiative, I wrote a PAL program to handle any project organized according to this method. The CPM technique, breaking objectives and projects into sequences of tasks to be accomplished, helped me organize my life and career.

Working for PTS exposed me to the operations of various types of businesses. A private partnership in the forestry industry, Cattermole-Trethewey, was seeking bankruptcy protection from its creditors. Along with two programmers and our supervisor, I was involved with writing the programs to resolve their insolvency issues.

Entrepreneurs willing to take business risks are an important factor in the growth and adjustment of a mostly capitalist economy to changing market conditions.

I was awarded a B.Sc. in Honours Mathematics, Class 1 at UBC's convocation in May, 1967. I intended to return to UBC in the fall to pursue a M.Sc. degree in Mathematics. My friendship with my classmates David Wei, Tony Dixon, and Len Horvath continued during our lives.

Summer of 1967 to April 1968.

In the spring of 1967, my brother worked for the District of Surrey, a large, rural municipality that was rapidly urbanizing. He convinced the Data Processing Manager, Mr. Reg Rhuwald, to offer me a summer job as a systems analyst / programmer. This meant an increase in my salary and the opportunity to work with my brother on some challenging applications.

Surrey's computer facilities were more sophisticated than those of PTS. They operated a Honeywell 200 computer with data stored on magnetic tapes. Four tape drives could be accessed by the main frame computer for input and output as data was processed. The initial data on paper, provided by the district's departments, were keypunched onto cards, read by a card reader, and transferred onto a magnetic tape by a program in the computer's memory.

Programs were invariably written in COBOL (Common Business-Oriented Language). The operating system also supported FORTRAN and RPG (Report Generating Language).

It was the responsibility of the Surrey's Assessment Department to estimate the probable selling price of every property in the district, including residential properties, vacant lots, farms, commercial and industrial structures. In 1967, they were also required to assess the value of machinery and equipment used by businesses.

The department clerks kept a register recording each property's owner and address. Each record was identified by an alphanumeric roll number, address, legal description, and subdistrict. The property register was sorted into roll number sequence for ease of access. The register was called the "Roll."

The District of Surrey was divided into subdistricts based on the area's common characteristics.

The record for residential properties included the building's current replacement cost, the number of bedrooms, floor space, basement, year built, carport, pool, other buildings, and for the lot, its size, foot frontage rate, and whether it was a corner or view lot. The record for a vacant property had the same data describing the lot's characteristics, plus its suitability for building a house.

The assessed values of farm, commercial, and industrial properties were based on the income they generated. The assessed values of residential and vacant properties were based on the sales of similar properties during the previous year. These assessed values were entered into the property's record, and used to calculate the owner's property tax.

With the district's rapid growth in population, construction of new houses, and increasing property values, it was no longer viable to maintain the Roll with a manual system.

Surrey's data processing department digitized the property register on paper onto a magnetic tape file. A program in the computer's memory inputted the data from the digital Roll and printed a notice of assessment for each property. These notices were sent to the registered property's owner.

Lists of all properties' assessed values were printed onto reams of computer paper in both roll number and property address sequences. These lists were available to the public. Owners could dispute their properties' assessed values.

When a property sold in Surrey, a document specifying the selling price, address, and legal description, plus the name and address of the purchaser, was sent to the assessment department. Clerks added the property's roll number and placed the sales record in a manual sales file with the other properties sold that year. The property's record in the property register was updated with the name and address of the new owner.

This procedure was computerized. The records in the manual sales file were uploaded onto a magnetic tape. By matching roll numbers, a property record on this tape was augmented with the information for the property on the digitized property register. The result was a digitized sales file identified by its year of sales. At the same time, the digitized property register was updated with the name and address of the new owner.

The appraisers of a city or municipal assessment department are professionals, trained to give an expert opinion of the probable selling price of a property. My task that summer was to write a program that would provide statistics on the change in property values in Surrey, and how these changing values depended on the location and type of property.

My program calculated the selling price to assessed value ratio by dividing the property's 1967 sales price by its 1967 assessed values. This change ratio reflected the property's change in value over one year, because the 1967 assessed values were based on 1966 sales.

An average ratio was calculated for all properties sold in each Surrey subdistrict. If the average ratio was greater than one, property values were increasing in the subdistrict; if the average ratio was less than one, they were decreasing. The change in property values could be investigated in larger regions by pooling these averages. Furthermore, these ratios could be averaged for different characteristics, like the year the house was built, floor space, basement, and the presence of a garage or car port.

The average 1967 assessed value for similar properties on the digital Roll could be calculated to identify properties that were over or under assessed in value.

The assessment department used this information to aid them in manual determining the assessed property values for the 1968 Roll.

When I returned to working for Surrey during the summer of 1968, I would write more programs to computerize the calculation of assessed values of properties in Surrey.

Over a number of years, these procedures would provide a recursive appraisal of property values with an automatic "regression to the mean" for similar properties identified by location and characteristics. On average, similar properties would pay the same property taxes.

In 1973 while a graduate student in economics at UBC, T. J. Wales and I investigated the proposition of equal taxes for similar properties. The District of Surrey kindly permitted me to write a program for their computer to extract the records for residential and vacant properties from their 1973 digitized sales file.

Public Finance theory assumes that taxes are capitalized. If buyers compare the physical features and taxes of houses, similar houses with lower taxes might sell for higher prices.

We used UBC's computer for our calculations. The results were published in the paper, "Capitalization of Residential Property Taxes: An Empirical Study" by T. J. Wales and E. G. Wiens (https://www.egwald.ca/wiens/elmerwienswalesresidentialtaxes.pdf). Our conclusion was that the evidence from Surrey did not support the hypothesis that residential property taxes are capitalized.

In 1975, Penny Gotzaman used the sales records I had obtained from Surrey for vacant properties to investigate whether taxes were capitalized on vacant lots. Prices for vacant lots in Surrey were rising circa 1973. A short-term investor might take the current year's taxes into consideration in calculating the expected profit from the lot's purchase and subsequent sale.

Gotzaman's results appear in Carleton Economic Papers, "Capitalization of Property Taxes: Vacant Properties by P. G. Gotzaman and E. G. Wiens (https://www.egwald.ca/wiens/elmerwiensgotzamanvacanttaxes.pdf). Her conclusion was that, while present, the degree of capitalization on vacant lots is limited.

In the fall of 1967, I returned to UBC as a graduate student in mathematics. I took courses given by four distinguished professors: Charlotte Froese—Numerical Analysis, Patrick Fischer—Computer Science, Rodrigo Restrepo—Theory of Games and Programming, and Maurice Sion—Real Analysis. These courses formed the basis of my M.Sc. Thesis in Mathematics that I wrote the next year.

Summer of 1968 to April 1969.

In May 1968 I returned to work for the District of Surrey and the staff had planned several programming projects for me. My first task was to computerize the production of the 1969 Roll. The data from my previous summer's programming had been used to produce the 1968 Roll. However, the Assessment Department's staff had completed this job manually.

All properties in Surrey were increasing in value due to its growing economy and influx of new residents. As described previously, the rate of increase in a properties' value depended on its characteristics and location. My program could calculate the selling price to assessed value ratio called the market change ratio for every property sold in any given year. The market change ratios could be averaged over subdistricts to create a table of average market change ratios per subdistrict.

The professional appraisers from the assessment departments could approve or alter each subdistrict's average change ratio. A property's assessed value from the 1968 Roll could be multiplied by the average change ratio of its subdistrict location from the table to generate its assessed value for the 1969 Roll.

This process was computerized and implemented the next year for the 1969 Roll.

Professional property appraisers are knowledgeable about statistics. From my undergraduate statistics courses, I knew of the technique called multiple regression. Appraisers actually use this technique when they estimate the probable selling price of a house based on its characteristics and location. They assign market weights to the characteristic variables of a house and lot, like floor space, number of bedrooms, basement, lot size, etc. They sum each market weight multiplied by its characteristic variable to yield an appraisal estimate of the selling price of a house and lot. This total can be compared to the actual selling price of the improved properties to verify the accuracy of the weights. This procedure can be repeated until an acceptable set of market weights is attained.

Multiple regression can replicate this process by using a computer program to generate market weights from the data of the assessment department's digitized sales. In 1968, Surrey's computer facilities based on the COBOL programming language were inadequate to implement multiple regression.

Surrey's engineering department was interested in using the computer with a view to reducing congestion at intersections and encouraging the flow of traffic along specific corridors. I was familiar with the mathematical technique of linear programming, which can be used to analyze traffic networks. Surrey's computer system supported FORTRAN, a language in my programming repertoire. However, in 1968 the computer's memory capacity was too limited to handle this application.

The other programming projects I worked on were mostly straight forward applications such as computerizing the voters list.

In the fall of 1968, I returned to UBC to complete my final year as a graduate student in mathematics. I took courses from two eminent professors: Douglas Derry—Point Set Topology, and Maurice Sion—Measure Theory and Integration. What I learned from the six mathematical courses as a graduate student was very helpful when I wrote my Ph.D. thesis in Economics in 1975, "Money as a Transaction Technology: A Game-Theoretic Approach" (https://www.egwald.ca/wiens/elmerwiensphdthesis.pdf).

During the 1968-69 academic year I taught a third-year UBC mathematics course, "Calculus and Analytic Geometry," based on the textbook by George B. Thomas, Jr. of the Massachusetts Institute of

Technology. Dr. Rodrigo A. Restrepo supervised my M.Sc. thesis, "Reduction of Games Using Dominant Strategies" (https://www.egwald.ca/wiens/elmerwiensmscthesis.pdf). I received my M.Sc. in Mathematics In the spring of 1969.

After graduation, I wanted a job as an Operations Research Analyst. Dr. Charlotte Froese gave me the book *Mathematical Methods of Operations Research* by Thomas L. Saaty and Dr. Rodrigo Restrepo *Elements of Queueing Theory* by the same author. I intended to read these books during my first ever extended summer vacation.

Spring and summer of 1969.

By April 1969, I had completed the requirements for a M.Sc. degree. Before leaving for Europe in June, I took a two-month job at Western Canada Steel in Vancouver. Their computer system was identical to Surrey's. I immediately started writing programmes in COBOL for the Accounting Department. These applications were fairly straightforward.

Western Canada Steel operated what is known as a mini-mill. It melted recycled scrap steel and pig iron in an electric-arc furnace to produce molten steel. The plant's engineers had a scientific computer with linear programming software. After testing the molten steel's properties, they computed the least-cost combination of alloys and quantities needed to meet their customers' specifications before further refining the molten steel.

In early June, my wife and I flew to Europe on a UBC student charter landing in London, England. We proceeded to Trafalgar Square and booked a room in a moderately priced, small hotel near the Paddington subway station. We enjoyed the sights of London staying long enough to watch the Trooping of Colour spectacle.

The Night Ferry from London took us to continental Europe. After one night in Brussels, we took a tram to its outskirts. We backpacked Europe and hitched a ride with a kind truckdriver going directly to Aachen Germany. He dropped us at the border and we crossed the Grenze into Aachen. I speak German and easily found a room with breakfast in a Pension.

We ate supper at a nearby Gasthaus. I ordered the Wiener Schnitzel mit Pilsen und Bratkartoffeln, und ein Bier. It was delicious. The next morning, I was introduced to the Continental breakfast, basically coffee, toast and croissants. I was used to having coffee, toast, fried eggs and perhaps bacon or sausages. I usually looked for a street kiosk or café for my daily morning protein fix.

For three weeks we travelled in Germany from Aachen to Munich, stopping frequently along the way to enjoy the historical sights, art galleries, museums, parks, zoos, and cuisine. Some highlights were Cologne Cathedral, castles along the Rhine River, Koblenz and Moselle wine, Heidelberg University, Schloss Heidelberg, drinking Weiss beer with Günter Grass in a Stuttgart Kneipe (pub), and lunch and beer in Munich's Hirschgarten.

When we arrived in Munich near München Hauptbahnhof, a German gentleman directed us to a nearby Pension. We had a large, clean room with breakfast at a reasonable rate. For five days we enjoyed the vibrant atmosphere of central Munich, before leaving for Austria.

Since our plan was to get to Athens, Greece by early July, we stayed in Salzburg for only one day and two nights in a rather grim Pension. The highlight of Salzburg was lunch in the Ratskeller, the restaurant in the basement of City Hall. Their famous Schweinsbraten (roast pork) met my expectations.

We intended to travel directly to Wien (Vienna) from Salzburg. A German gentleman gave us a lift in his high-powered car and showed us Austria's picturesque countryside and Lake District before dropping us off in Linz in the late afternoon. One night in Linz and we were off to Wien.

Pensions are ubiquitous in European cities. In Vienna, we had a large room with a high ceiling in an older building. On our first evening, we crossed the street to a pub for a light supper. I was surprised to see the patrons transfixed by the cowboy show Bonanza on TVs distributed about the room. I had a beer with bratwurst and sauerkraut, fitting in with the crowd with my command of the German language.

There is so much for tourists to do in Vienna. An absolute must see is the Imperial Schönbrunn Palace, gardens, and zoo. We saw only a fraction of its attractions during one long, tiring day. I will never forget watching the Lipizzaner horses during their daily training. We took the train to the Vienna Woods, but were unimpressed given our British Columbian background. Unfortunately, we did not take in the Albertina Museum, a performance at the State Opera House, or a tour of the Vienna City Hall. However, we appreciated the many stately buildings as we wandered daily around the city.

Vienna is about 800 miles away from Athens, Greece. Created in 1883, the Orient Express rail line connected the major cities of Europe with Istanbul, including a line to Athens. We boarded the train in Vienna and after passing through Graz we arrived at the Yugoslavia border. Here, the electric-powered locomotive was switched-out with a coal burning, steam locomotive.

Lynn and I shared a compartment with four travellers. The train proceeded through Zagreb and Beograd. Half-way to Skopje, the passenger cars at the back of the train were disconnected. A friendly conductor told me that these passenger cars were on their way to Istanbul. At the Greek border, the Yugoslav locomotive was switched-out with a diesel-powered Greek locomotive.

The route to Athens passes along the coast of the stunningly blue Aegean Sea. While sharing this vista, my wife and I chatted with an English couple in our compartment. Dick and Sue were travelling by train from London to Athens. They mentioned that they planned to tour the Greek islands after a few days in Athens.

My wife and I disembarked at the Attiki Metro Station within walking distance of central Athens. At the station, a representative from a hotel offered us a room at a reasonable rate. For five days we stayed in the recently built Attiki hotel in a spotless room with a private toilet and bath. Our hotel was actually about two miles from Constitution (Syntagma) Square at the center of Athens. We usually took the subway in to take in the nearby attractions.

A military junta backed by the USA ruled Greece in 1969. The Greek Parliament building across Amalias Avenue is located on the north-western part of the Royal Garden. The military guards gave us a friendly greeting when we walked into the garden. As we headed south, the Parthenon came into view in the south-west.

Over the next four days we visited the Temple of Zeus, the Parthenon, the Theatre of Dionysus, and the Ancient Agora of Athens. In between, we watched the tourists and locals as we drank Greek coffee, and looked for places for lunch and supper.

While in high school, I enjoyed reading Greek Mythology and Homer's poetic account of the Trojan war in the Iliad and Odyssey. When I visited a travel agency about a trip to the Greek Islands, I chose the Island of Ios from the list the English gentleman recommended. According to some accounts, Homer died on Ios. Two days later we checked out of our hotel and took the subway to Piraeus. There we boarded a ferry serving the Greek Islands. We were surprised to encounter Dick and Sue, the English couple we met on the train to Greece, who were also on their way to Ios.

When we disembarked at Gialos, the port of Ios, they scurried west along the beach towards their accommodation. I got us a room in a small hotel next to the harbour. When we met Dick and Sue for breakfast in the hotel's harbour patio, they suggested we take a room in their hillside villa hotel.

My wife and I stayed in the rudimentary villa overlooking Gialos for ten days. Most evenings, the four of us walked to the harbour and up the hill to Chora, the capital of Ios. We had long conversations over supper sitting in a restaurant's outdoor patio on the village's central square. The menu listed stuffed tomatoes, moussaka, Greek salads with feta, fish, souvlaki, roasted potatoes and chicken. With red or white wine, our meals were accompanied by Greek music. I like Retsina, a special wine flavored with pine resin from the Aleppo tree.

Late in the evening, we carefully stepped along the dark, cobble-stone road down to the harbour. If the café was open, Dick would persuade us to have a dish of Greek yoghurt with espresso and maybe a glass of ouzo. We would often arrive back at our rooms well after midnight. Dick and Sue invited us to look them up in London on our way back to Canada.

We remained on Ios for a few more days after Dick and Sue left for the Island of Mykonos. During the day I explored the area around Ios Bay. The white washed Church of Agia Irini stood south across the bay from our villa. I took the path to the church and continued along the shore until I reached a small, isolated cove. I skinny-dipped here on a couple of afternoons until I was told the Mediterranean Sea supports many species of sharks.

On our return to Athens, we were fortunate to get a room in the same Attiki Hotel. On the morning of July 21, we took the subway to Constitution Square. The Square's kiosks displayed newspapers announcing the first landing on the Moon. Half-way through our European tour, we were at our furthest distance from London. It was time to head back by way of Italy.

Late July is the height of the tourist season. We became anxious when tickets to Italy were not available from a couple of agencies. My friendly English travel agent, however, managed to get us two tickets on a boat that would take us from Piraeus to Corfu and on to Taranto, Italy. However, he warned me it was not designed for tourists.

The boat transported both cargo and passengers. Some families with large amounts of baggage occupied parts of the deck. Our tickets allowed us to sit on chairs in the passenger section and go on deck to watch the shoreline. We had brought along enough food and water for two days of travel.

After leaving Piraeus, the boat squeezed through the Corinth Canal between the Greek mainland and Peloponnese peninsula to the Ionian Sea. It was late in the evening when the boat docked at Corfu's Old Port. It would leave for Italy early in the morning, so we stayed on board for the night.

Our tickets entitled my wife and I to hammocks in separate women and men's quarters. I shared a large room with a dozen swarthy Greek men. I had my first experience with a primitive squatting toilet and spent a restless night listening to the snoring. My wife had a similar experience. The next time we would sleep in the passenger section.

At Taranto, we caught an overnight train to Termini Station in the heart of Rome. We found a nice room in a nearby family-oriented hotel. Its basement restaurant offered reasonably priced breakfast, lunch and supper meals. The concierge permitted us to leave our bags until our room was ready. We had arrived in another "eternal city". It was now time for some serious sightseeing.

On our first afternoon, we wandered to the Trevi Fountain and tossed coins in the appropriate manner. The panoramic view at the top of the Spanish Steps familiarized us with Rome's layout, and we returned to the hotel too tired for supper.

It was stimulating to see the sites I had only read about in ancient history and literature high school and university courses. During our five days in Rome, we leisurely absorbed the Colosseum, Trajan Column, Roman Forum, and the Pantheon. It took one day to take in St. Peters Basilica and the Sistine Chapel in the Vatican City. The paintings on the walls and ceiling of the Chapel are incredible. When I took a UBC Art History course in 2013-14, I gained a deeper appreciation for these paintings by Michelangelo, Sandro Botticelli and others.

I rented a car to facilitate our trip from Rome to Innsbruck, Austria. We enjoyed the scenic country side on the way to Florence (Firenze). After parking near the Ponte Vecchio, we crossed the "Old Bridge" and ate a light café lunch. An easy walk brought us to the Accademia Gallery where we admired Michelangelo's Statue of David along with a throng of tourists. On the way back, we gazed at the exquisite jewelry in the shops along the closed bridge.

I wanted to get to Genoa. We stopped briefly to view the gravity defying Leaning Tower of Pisa. The Tower, Pisa Cathedral, and the Pisa Baptistery are marvels of medieval architecture.

The highway to Genoa runs parallel to the coastline. I exited the highway near Genoa to take the road along the coast. We arrived in the evening and checked into a hotel that looked like a villa. The proprietors spoke only Italian, and the next morning I had difficulty settling the bill for our supper and room. After some negotiation, my offer for twenty dollars was accepted.

Sometime in the future, I want to take an extended holiday in the lovely harbour city of Genoa and enjoy its cuisine, historical buildings and vibrant old town. Before we left for Venice, we had an Italian breakfast from a kiosk of coffee and pastry at the Monument to Christopher Columbus.

It is a four-hour drive across northern Italy from Genoa to Venice. After checking into a hotel in mainland Mestre, we left our car and took a tram to the Island of Venice. Venice was crowded with afternoon tourists. It has a lot of old buildings, canals, and gondolas. We restricted our visit to walking around the iconic St Mark's Square. Before catching the tram back to the hotel, we savoured a couple of colorful gelato cones from a vendor.

From Mestre we drove north through the southern Alps to Cortina. We arrived in time for lunch. I was aware, perhaps from CBS and CBC newscasts, that the 1956 Winter Olympics were held in Cortina. The ski runs could be seen on the sloping sides of the spectacular mountains at the edge of the city.

The Highway leading north from Cortina joins the main highway from Trento to Innsbruck south of the Brenner Pass. The elevated roadway runs through the Pass over green alpine valleys where dairy cattle graze in the summer.

Innsbruck is located in a valley surrounded by the Austrian Alps. We discovered a cozy hotel situated amongst a group of hotels and eateries in the downtown area. I dropped off my car at the rental agency and after some haggling closed my account. The facilities from the 1964 Winter Olympics were still accessible to visitors. We rode the cable car to the summit of one of the Olympic ski slopes.

After a few days we left for Switzerland. We hitched some rides, passed through Liechtenstein, and eventually arrived in a small, pretty remote community on Lake Geneva. Although it was August, we got a room in a pleasant Gasthaus at a reasonable rate. I needed a few days to recover from my stomach virus. This was the only time when either my wife or I fell ill during our journey.

With the information provided by the concierge, we took a bus to the main railway station in Geneva. Our itinerary was flexible. We were making it up as we went. Before our flight to Canada, however, we wanted to spend ten days in London.

From this point, we continued our journey by train. After lunch, we boarded the next train to Basel. At the huge SSB Station, once again we had to find a suitable hotel. An agent at the Visitor Center booked us a room in a hotel near the Basel Zoo. Basel is located at the tripoint where the Switzerland border meets the German-Switzerland and France-German borders. North of Basel, the Rhine River forms the France-German border. East of Basel, it forms the Switzerland-German border. Basel's culture and food are influenced by French, German, and, to a lesser extent, Italian.

Zoo Basel is located in a large park at the city's center. We spent two pleasant afternoons wandering throughout the zoo, looking at displays featuring exotic animals and birds. Basel's university is comparable in age to Cambridge and Heidelberg universities. I had received my M.Sc. in Mathematics in the spring. I was considering enrolling in a Ph.D. program in Mathematics and thought it would be enjoyable to live in either Basel or Heidelberg.

We had less than three weeks left on our trip. The train from Basel via Koblenz to Amsterdam ran along the Rhine River for the first leg. We arrived at noon at the Amsterdam Centraal Station. Amsterdam was packed with tourists. The Visitor Center's agent said the hotels were booked. But if we returned later in the afternoon, she would make some calls to find us a room.

While we waited, I got a haircut at the station's barber shop. After months of travel, I looked like a hippie. The barber immediately shortened the hair on one side. When I suggested in German that it was rather short, he replied, "Bist du nicht hier, um zur Armee zu gehen?" What could I do, but let him finish the job. How did he know I am of Frisian descent?

Meanwhile, the agent had found us a room, at the agreed rate, in a private house a few canals over from the "Old Centre." Our kindly, Dutch landlady showed us our room with the toilet and bath down a short hallway. We sampled a couple of local beer with our supper in an Amsterdam "brown café," just around the corner. After taking a bath, we crashed until noon the next day.

With so many historical sites in the cities center, Amsterdam is perfect for tourists. In four days, we took in as much as we could. The Rijksmuseum displays paintings by Van Goh, Vermeer, and Rembrandt van Rijn. Of course we had to see the Rembrandt's "The Night Watch," the most famous painting by the Dutch Masters. I found it interesting that Rembrandt's wife, Saskia, was a Mennonite. He was commissioned to paint numerous portraits of wealthy Amsterdam Mennonites, the best-known being "The Mennonite Preacher Anslo and his Wife" located in the Staatlich Museen zu Berlin.

University friends recommended visiting the Heineken Brewery. Following our landlady's suggestion, we arrived for an early morning tour at the huge Heineken Brouwery building in the center of Amsterdam. A brewery has many vats with gauges, connecting pipes, and conveyer belts carrying cans and bottles of beer. At the end of the tour our group was offered beer and sandwiches. It was still morning, and I could drink only one glass of beer.

The Dutch colonized Indonesia by way of the Dutch East India Company until 1799, and then by the Dutch Government's direct rule until 1949. Their control was interrupted by France and Britain, 1806-1816, and by Japan during WWII. During the 1950s, my hometown of Yarrow, B.C. welcomed waves of immigrants, Dutch Colonists leaving Indonesia, Germans and Ukrainians fleeing the aftermath of WWII, and Hungarians displaced by the 1956 revolution.

The Indo-Dutch Rijsttafel feast is a remnant from Hollands colonial past. My wife and I were seated at a table for two at an "Indies" Restaurant. After bringing us tea, the server suggested we have the white rice with fifteen side dishes. We were served bowls of rice and a few rijsttafel dishes. We began eating, leisurely sampling the dishes with a variety of ingredients as they arrived. When we had eaten our fill an hour later, there were still some untouched dishes on our table.

One day we wandered around Vondel Park and browsed the vintage and second-hand items at a flea market without buying anything. In the evening, we walked through Amsterdam's famous Red-Light District. Tourists crowding the narrow sidewalks along the canals were offered entertainment in night clubs and casinos. There were ladies sitting in windows, sex stores, peep shows, bars and cafes.

In 1969 it was possible to take a Boat Train from Holland to England. We took the train from Amsterdam to Rotterdam in the afternoon, then travelled by ferry to Harwich, England overnight, and arrived on the train to London in the mid-morning. By early afternoon, we had a room in the same Paddington hotel where we stayed in June.

After settling into our room, we walked the short distance to historic Hyde Park. The pathway through the park took us along Serpentine Lake. At Hyde Park Corner, we took the tube back to our hotel. We saved walking by Buckingham Palace for another day. After a late afternoon nap, we ate a traditional English meal of fish and chips at a nearby pub.

I wanted to see Cambridge University. London King's Cross Railway Station is a short tube ride from Paddington. It is a two-hour train trip to the Cambridge Train Station, and a half hour walk to the Cambridge University Press Bookshop. I bought two books, Bertrand Russell's "History of Western Philosophy" and Maurice Sion's "Methods of Real Analysis." Professor Sion's book is based on his UBC

Real Analysis course lectures that I took as a mathematics graduate student. Whenever I visit a university, I enjoy perusing the bookshop.

With our mode of travel, hitchhiking, buses, trams, boats, and trains, shopping and carrying our purchases while on the continent was difficult. Our bags were always full. I bought a large, leather suitcase to hold the items we bought in London. The stores on Oxford Street offer a wide variety of women's and men's clothes. My wife bought some dresses and I bought a three-piece suit. We bought shirts, blouses, and other goods at one of the many Marks and Spencer's stores in London.

When in London, one must see the West End Shows. On two consecutive days, we watched performances of Agatha Christie's, "The Mouse Trap," at the Ambassadors Theatre and "Hair" at the Shaftesbury Theatre. After the performances, we enjoyed early suppers and hanging out with other tourists in the public squares. Many years later, my Vancouver Actors' Theatre Company presented my play, "Critical Paths," at the 1994 Vancouver Fringe Theatre Festival (https://www.egwald.ca/ubcstudent/theatre/criticalpaths.php). In 2013, two of my dramas, "Glory Road" and "Canon in D Major," were presented on UBC's CiTR Radio (https://www.egwald.ca/ubcstudent/plays/index.php).

I like visiting zoos. In 1969, the Snowdon Aviary was part of the London Zoo situated south of Regent's Park. I recall seeing many African animals, including tigers, lions and gorillas. The Aquarium was stocked with freshwater and saltwater coral fish. The Aviary, which contained exotic birds, has since been redesigned as Monkey Valley, home to white colobus monkeys.

With four days left in our vacation, we saw the Crown Jewels in the Tower of London and watched the Changing of the Guard at Buckingham Palace. From a nearby iconic red telephone box, I called Dick and Sue, the friends we met in Greece. They were happy to hear from us and recommended that we spend the final three nights with them before we left for Canada. The following day, we were invited to their home for supper.

Late the next morning, we checked out of our Paddington hotel. The concierge stored our bags until it was time to leave for Dick and Sue's. We had a leisurely lunch at a nearby pub. In the mid-afternoon, we took the tube to Brixton south of the Thames River. Working class South London is different from cosmopolitan North London, reflecting persistent, historical socioeconomic factors. Dick and Sure lived in a suburban London borough about five kilometers south of the Thames River.

We took a black cab to Dick and Sue's after exploring Brixton's market, bar, and café area. They were waiting for us at their home in a hilly, pleasant residential area. A bathroom separated our second-floor bedroom from theirs. After freshening up, we joined them for supper. Sue offered us cottage pie along with the red wine we brought. Over dessert and coffee, we resumed conversations we shared on Ios. Given their work constraints, we discussed activities we might do together for the next two days.

The next day was Saturday and Sue's flower shop was open. Very early the next morning, we accompanied her to the New Covent Garden Market to buy fresh flowers. After dropping Sue and the flowers at her shop, Dick, Lynn and I ate breakfast at a café. Dick had arranged to meet his dad later in the morning at the Old Vic Theatre. As the Property Manager, he was responsible for all theatre operations except play production.

George Bernard Shaw's play, "Back to Methuselah," was playing that summer. Dick's dad showed us the props, sets, scenery, and technical equipment required to perform the play. The theatre's Wardrobe Mistress explained her role ensuring that costumes and props are ready before and during performances. She showed us costumes from past plays arranged in long rows in the wardrobe area, including the military uniform Laurence Olivier wore as Edgar in "The Dance of Death."

Dick, Lynn and I met Sue at her shop in the afternoon. We had a snack before catching the early dog races at Wimbledon Greyhound Stadium. Every dog entered in a race is listed on a Greyhound Form Chart with statistics describing the dog's time and finishing position in recent races. With a M.Sc. in Mathematics, I can detect patterns others might not see.

After the first race was over, I checked the chart to see if the result was predictable. To win an exacta bet, one must choose the first and second dogs in the race in the correct sequence. Given the likelihood of doing this, a small bet can yield a large return. I perused the form for the next race, and placed an exacta wager on two dogs. To our surprise, I won!

Dog races are run every fifteen minutes. Nothing caught my attention in the next two or three races. As I recall, I won two more wagers, an exacta and a show. When I stopped betting, I was up forty pounds. As planned, we took Dick and Sue to a restaurant for a wine and steak dinner. I paid for our meals with the money I made gambling.

On Sunday morning, the last day of August, we drove to Brighton. The Palace Pier was surrounded by beaches filled with sunbathers. Our lunch at a Pier Café of fish and chips was delicious. On our way back to London, we stopped at the village of Ditching, known for its arts and crafts.

On Sunday night, Sue invited us to her parent's pub which was located south of the Thames River in the Bankside locality. We enjoyed a steak and kidney pie with unlimited glasses of beer and I told them how much enjoyed this traditional English meal. Sue's father told us that their pub had the deepest and coolest beer cellar of any pub in South London.

During the evening, we watched the pub patrons playing darts. Sue explained that many of the pub regulars worked on the Thames docks and lived in the neighbourhood. Additionally, she informed us that, some years ago, a local gang briefly stored a bag of money in the pub's safe, warning her parents not to examine its contents.

After the pub closed, Sue's parents invited us to stay for a private party. Bowls of jellied eels, cockles, winkles, welks and seafood sauces were placed on tables. They encouraged us to sample these seafood delights with more beer and various types of whiskey. Since our flight to Canada was not leaving until Monday evening, we could stay late and enjoy the party.

Monday, September 1, 1969 was a bank holiday in England. In the afternoon, we packed our bags into the boot of Dick and Sue's car and drove to Heathrow Airport. Our route took us over several bridges with great views of the Thames River. We were taking a charter flight and advised to arrive early.

The sun was setting when the 707-jet liner left Heathrow. For about eleven hours, we flew into the sunset, arriving at Vancouver Airport at 8pm on Labour Day. We stayed with my wife's family for several weeks while I sorted out my job prospects.

In the summer of 1969, Europe was experiencing the aftermath of the previous year's widespread protests. These social movements resulted in shifts in the political landscape. Voters in Germany and Austria switched from center-right to center-left political parties. Support for Italy's centrist party was eroding. Yugoslavia was a communist dictatorship. The right-wing military junta aligned Greece with the USA. In England, Harold Wilson's governing Labour Party would be replaced by Edward Heath's Conservative Party in 1970.

Following the Second World War, the capitalistic democracies of Western Europe rebuilt their economies with state-owned enterprises (SOEs) in crucial industries. As examples, Britain and Germany's railroads, British Rail and Deutsche Bundesbahn, were state owned. The oil and gas company, Ente Nazionale Idrocarburi, had a monopoly on oil and gas exploration in Italy. In socialist Eastern Europe, most of the countries' GDP (Gross National Product) was produced by SOEs.

About a decade later, I taught economics and operations research at Carleton University in Ottawa. Richard Harris and I had a research grant to investigate the role of the SOE, Petro Canada, in the Canadian oil and gas industry. We published our results in a series of papers beginning with the Carleton Economic paper, "Behaviour of a Government Firm in Oligopolistic Situations" (https://www.egwald.ca/wiens/elmerwiensgovfirmbehaviour.pdf).

September 1979 to October 1970.

While I was completing my M.Sc. studies, I was interviewed for a job as an Operations Research Analyst with McDonnell Douglas Aircraft Company in St. Louis. I was utterly amazed by their facilities, including a McDonnell Aircraft Project Mercury capsule. I tentatively accepted the job they offered with the understanding we would finalize the arrangement when my wife and I returned from Europe in September.

However, McDonnell Douglas lost a military contract that the summer, making their job offer problematic. In September, I applied for operations research jobs at Bell Telephone Company, Montreal and Ontario Hydro, Toronto. After my interviews, I accepted the job at the Research Division of Ontario Hydro. My wife and I moved to Toronto in October, 1969.

My wife found us an apartment on La Rose Avenue in Etobicoke, Toronto. In order to get to work, I caught a bus to the Royal York Subway Station, the subway to Kipling Station, and another bus to the Ontario Hydro's Research Division at 800 Kipling Ave. In the Operations Research (OR) group, I was the only mathematician among six engineers. The Research Division created and evaluated methods to enhance power production, transmission, and distribution.

Our OR group built mathematical models to optimize Hydro's Power system. Basically, we used models like linear programming to determine how to supply the demand for electricity while minimizing its cost of production. Using the concept of the Economic Order Quantity (EOQ), our models helped Hydro manage the inventories of thousands of items it uses in its operations. My UBC courses and summer work enabled me to contribute to these models.

Because his skills were needed elsewhere on a major Hydro project, the OR groups manager (who had a Ph.D. in engineering) was replaced with a bachelor's degree engineer. My friend Brian Isbister (with a master's degree in engineering) and another disgruntled engineer quit their jobs to join other Hydro divisions. Consequently, the work environment at Hydro became uncomfortable for me. When I attended the 1970 Canadian Operations Research Society annual convention, the Operations Research Manager at Canada Packers headquarters in Toronto offered me a position.

I spent four summers working in food processing. Canada Packers had plants in BC and across Canada. I believed that working for Canada Packers would further my career. Even though the job offer was at a higher salary, I am not sure that I made the right decision.

A public utility like Ontario Hydro generates electricity at lowest cost for its customers. Hydro's monopoly prevents its customers from buying electricity elsewhere. A private company like Canada Packers (CP) produces products for its customers in order to earn a profit and distribute dividends to share holders. However, they had to compete for customers with other food processors like Burns Foods, Swift Canadian, and Maple Leaf Foods. In 1970, CP's sales revenue was twice that of its two major competitors.

CP was an amazing company (History of Canada Packers: Encyclopedia.com https://www.encyclopedia.com/books/politics-and-business-magazines/canada-packers-inc). From its headquarters at 30 St. Clair Avenue West in downtown Toronto, CP's managers ran a diversified company with 20,000 employees and annual sales revenues of one billion dollars. It had numerous meat packing plants and facilities producing feed and fertilizer, consumer goods, dairy products, and pharmaceuticals. Over the years, Toronto's CP abattoir butchered tens of millions of pigs earning the city the moniker of "Hogtown" (http://www.trainweb.org/oldtimetrains/stockyards/stock.htm).

CP products, including meat (Maple Leaf brand), fruit and vegetables products (York Farms brand), and edible oils like margarine and salad dressings, were sold to customers across Canada by supermarket chains and independent grocery stores. In order to handle inventories, product mix and sales forecasting, CP maintained computer data processing centers in Toronto, Montreal, Winnipeg, and Edmonton. The responsibility of our OR group was to support CP's managers in applying Operations Research methods with these computers and / or with rule-of-thumb heuristics.

The Operations Research (OR) group consisted of our Chief, Dr. "Rama" Atluru, George Dalmeny, Allan Robertson, Ann-Marie, a summer student, and myself. Ann-Marie helped Dr. Atluru implement linear programming models so that the most profitable types of salad dressings, and retail cuts from a side of beef were produced. George liaised with the Prairie meat packing plants. Allan worked with the Grocery Division. I was assigned to CP's huge plant at the corner of Keele Street and St. Clair Avenue.

Canadian Pacific and Canadian National Railways' stock cars carried livestock for butchering, and tanker cars carried vegetable oils for refining to Toronto's CP plant. The plant managers forecasted their customers' needs from CP's mix of products. The supplies (inputs) of livestock and vegetable oils had to arrive in sufficient quantities and at the right time to produce the goods (outputs) customers demanded. It was the responsibility of CP workers, known as traders, to balance inputs and outputs for maximum profit over time.

At the Toronto plant, twenty-four traders were seated at desks in a matrix grid consisting of four submatrix grids. Groups of six traders, representing CP divisions, were allocated to the four sections of the trading floor. A trading manger supervised the traders in each section. The trading floor was the nerve center of CP's operations. This was CP's Toronto profit center, decentralized by division. I abstracted CP's traders' buying and selling activities as a "Technology of Exchange" in my 1975 Ph.D.

thesis, “Money as a Transaction Technology: A Game-Theoretic Approach” (https://www.egwald.ca/wiens/elmerwiensphdthesis.pdf).

Sales can be predicted statistically using operations research techniques such as time-series analysis and modelling the factors that influence market conditions. By streamlining a plant’s manufacturing process and ensuring a ready supply of inputs, sales forecasting and inventory control of both inputs and outputs can lower costs, boost sales and revenues, and increase profits. I used my experience with inventory control at Ontario Hydro, knowledge of business statistics, and regression analysis to assist the traders in the Edible Oils group at the Toronto plant.

The traders quickly discovered how to modify their rule-of-thumb heuristics by using these OR methods. Traders at other CP plants as well as traders in the different areas of the trading floor were given access to these forecasting approaches. After a few months, I concluded that the traders' operations would not be improved by further mathematization.

Rather, I started to consider whether financial rewards for traders and their boss could increase sales, revenues, and profits. Some years later, I included managerial economics and incentives in my Operations Research course at Carleton University in Ottawa. My conclusions on this topic are posted on a series of webpages at: https://www.egwald.ca/economics/econpage.php. I demonstrate that a bonus scheme that uses the appropriate positive weights for profits and revenues may simultaneously balance profits with sales growth, and minimize the costs of production.

Canada Packers was a well-run company with ethical managers, directors and owners. It aggressively expanded its operations locally and internationally by way of investments, acquisitions, mergers, output and input diversification. New technologies were incorporated when profitable. When four of its major competitors pleaded guilty to charges of price fixing in 1983, Canada Packers was exonerated.

October 1970 to August 1972.

In October 1970, my wife and I returned to Vancouver. I was 25 years old. I rejoined Surrey’s Data Processing Department as a systems analyst / programmer. The systems I created with Surrey's Assessment Department over the summers of 1967 and 1968 were being used to provide data processing services to Assessment Departments in nearby municipalities. They paid Surrey for its sophisticated IT capabilities while they used Surrey as a blueprint to build their own Data Processing Departments, complete with Surrey’s systems and programs.

Surrey’s Honeywell 200 computer system was upgraded from a tape data storage system to a hard disc storage system, with a faster central processor. These improvements allowed us to create increasingly complex computer programs by speeding up the computer's execution of production tasks. We experimented with implementing linear regression for the assessment department, and linear programming for the engineering department. However, this required skilled staff members in these departments to comprehend and use these methods.

To eliminate inequities in property taxes among homeowners, British Columbia’s Social Government wanted to standardize how cities and municipalities, local jurisdictions, assessed properties. Furthermore, the rapid rise in property values in BC resulted in an overall rapid year-to-year rise in assessed values. For political motives, the government legislated that the total assessed values of all properties for taxation purposes in a jurisdiction could not increase by more than five percent over the total assessed values from the previous year. However, market assessed values on individual properties within a jurisdiction were expected to reflect comparable real market prices.

The program I wrote in 1969 generated a property's market assessed value by multiplying its previous year's market assessed value by the average market change ratio of its Surrey subdistrict location. However, newer houses in upscale neighbourhoods or those with a view were increasing in value more rapidly than other houses in the subdistrict. This program was modified to account for such differences in a house's characteristics and location.

The market assessed value for every property was calculated for the 1971 Roll, and placed in the property's record in the digitized property register on the hard disc storage system. The total assessed value for 1971, Tot_1971 was compared to the total assessed value for 1970, Tot_1970. The cut-back-factor, f, was calculated as: f = 1.05 x Tot_1970 / Tot_1971. For each property, the cut-back-factor, f, was multiplied by the market assessed value to yield the property's assessed value for taxation. Political sleight-of-hand! Without the legislation, the same amount of property taxes could be collected but at a lower tax rate. BC's W.A.C. Bennett's Social Credit government was replaced by Dave Barrett's NDP government on September 15, 1972.

Along with my work at the Assessment department, I wrote computer programs and / or devised computer systems for Surrey's Accounting, Welfare, Permits and Licences, City Clerk, and the Water Works departments. I devised systems and wrote programs that computerized Surrey School District's accounting system. I also wrote programs for the Greater Vancouver Regional District (GVRD) computerized accounting system.

Willliam Vander Zalm was elected a Surrey Alderman in 1965, and the Mayor of Surrey in 1969. The amiable Mr. Vander Zalm had federal and provincial political ambitions. He keenly supported Surrey suppling data processing services to neighbouring municipalities, the GVRD, and the Surrey School District. During his tenure from 1975 to 1983 as a member of BC's Legislative Assembly, he served as Minister of Human Resources, Municipal Affairs, and Education. From 1986 to 1991, he was the Premier of British Columbia. Rita Johnston, Vander Zalm's political ally, was a member of Surrey Council from 1970 to 1983, and from 1983 to 1991 a member of BC's Legislative Assembly. When Vander Zalm resigned as Premier in 1991, she became Canada's first female Premier.

My interest in economics was kindled by working for Canada Customs and Immigration, a commercial data centre, a municipal government, a steel mini-mill, a huge public utility, and a private conglomerate. In the spring of 1971, my wife and I attended an evening seminar on economics given by Dr. J.D. Rae, a professor in UBC's Economics Department. He told the class that the growth of the economies of South Asian nations was largely dependent on foreign trade.

During UBC's 1971-72 Winter Session, I took evening courses in intermediate microeconomic and macroeconomic analysis. The textbooks were *The Price System and Resource Allocation* 4th edition by Richard H. Leftwich and *Macroeconomics: The Measurement, Analysis and Control of Aggregate Economic Activity* 4th edition by Thomas F. Dernburg and Duncan M. McDougal. The authors employ illustrations and basic mathematics to help their target audience of upper-level college students understand their explanations.

I really enjoyed these courses. The instructor Dr. Gopal J. Yadav, with the IMF for 33 years, encouraged me to apply for UBC's Masters program in Economics. I was accepted and in August 1972, resigned from my job in Surrey.

Meanwhile, I read H. W. Spiegel's *The Growth of Economics Thought* in the summer of 1972. He takes a cultural approach (non-technical) to examine how the intellectual atmosphere and political climate influenced authors to develop their timely economic concepts. His book introduced me to the doctrines of Adam Smith, T. R. Malthus, David Ricardo, John Stuart Mill, Karl Marx, Augustine Cournot, the Austrian School, Leon Walras, The Cambridge School, the Swedish School, John Maynard Keynes, and many others.

September 1972 to August 1975.

In 1972, my wife graduated with a bachelor's degree in education, and started teaching high school in the Delta School District. I was 27 years old. In September, I began UBC's Masters Program in Economics. The 1972-73 Winter Session was a very busy time for me. Economics graduate courses require a lot of reading. Most of my classmates had undergraduate or Masters degrees in economics. Many economics concepts needed to be grasped quickly. I leveraged my Operations Research business expertise and government work to anchor economic concepts in real-world situations.

The employees (and owners) of an enterprise hold a different perspective than academic economists.

When an economist examines an electrical public utility, for instance, their perspective is different from that of the utility's employees. Economists seek answers to questions such as whether the utility is a monopoly subject to government regulation, whether it charges excessive rates on outputs, whether it produces output at minimum cost, and whether its actions impact how the economy distributes its limited resources.

Under the guidelines set by the government, the management of Ontario Hydro worked to meet the demand for power. This demand differed over time, by region of the province, and by customer type. Christmas and summer saw higher peak loads. The rate structure distinguished domestic customers from commercial and industrial. The electricity power generating sources were not spatially distributed evenly with respect to the location of consumers. The power grid had to balance these disparities in demand and supply of electricity.

While economists look at a utility from the outside, utility staff look outside to fulfill their mandate.

Even though modern economic analysis uses mathematics, many authors prefer to examine an idea using words, leaving the more precise mathematical analysis in the appendix. In *Foundations of Economic Analysis* Paul Samuelson quotes Josiah Willard Gibbs, "Mathematics is a Language." In order to demonstrate that many economic topics may be modeled using comparatively basic mathematics, Samuelson integrates mathematics into his text. He asserts that writers who employ language to convey fundamentally mathematical ideas are practicing "mental gymnastics of a peculiarly depraved type."

Since UBC's graduate school economics necessitated mathematical expertise, my M.Sc. in mathematics was useful. I was able to comprehend and use methods like forecasting, game theory, utility theory, multiple regression, constrained maximization—profit maximization and cost minimization—and linear, nonlinear, and integer programming.

The economic models I had learned in Dr. R.A. Restrepo's graduate mathematics course in 1967–1968 were the most significant. Mathematical models of economies, such as the Arrow-Debreu Model of General Equilibrium, Leontief's Model, and Von Neumann's Model for an Expanding Economy in Equilibrium, provide the basis of most of advanced economic theory.

The idea of the total derivative of a function of multiple variables is helpful if economics is viewed as applied mathematics. It measures the total change in the function's output due to changes in all of its input variables.

In the Introduction above, an economic production function was presented as q = f(L, K, M), where q stood for output and L, K, and M stood for labour, capital, and materials and supplies, respectively.

The partial derivative of f with respect to (wrt) L, for example, is represented as $\partial f / \partial L$, or f_L, or f_1. It denotes the change in the output, q, due to a small change in the input, L, holding the values of K and M constant.

The total differential of f is written as the expression:

$$(1) \qquad dq = df = (\partial f / \partial L)\, dL + (\partial f / \partial K)\, dK + (\partial f / \partial M)\, dM,$$

where dL, for example, measures the increment in the quantity of labour. The variables dL, dK, and dM are free to change independently. The partial derivatives are evaluated at a point ($\underline{L}$, $\underline{K}$, $\underline{M}$).

The expression (1) says that we multiply each increment (dL, dK ,dM) from ($\underline{L}$, $\underline{K}$, $\underline{M}$) by its corresponding partial derivative and add to obtain the change in the output, dq.

The total differential (derivative) of the function f at the point ($\underline{L}$, $\underline{K}$, $\underline{M}$) is the best linear approximation near this point with wrt its variables L, K, and M.

G.C. Archibald and Richard G. Lipsey's *An Introduction to a Mathematical Treatment of Economics* introduces the concept of a total differential as well as numerous other mathematical ideas that are helpful in economic analysis.

Microeconomics.

1972-73. Microeconomics was taught by Professor G.C. Archibald in the fall and spring semesters of 1972–73. Understanding his lectures was aided by familiarity with his book with Lipsey. As I write this, I refer to my course notebooks, which contain the transcriptions of his lectures.

In Economics 500, Archibald modelled the demand and supply of products by consumers and firms. Businesses require (demand) inputs, like labour, in order to produce (supply) profitable products. Subject to financial limitations (budget constraints) that rely on their labour supply, consumers look for (demand) goods to consume.

Two examples are provided here to exemplify economic models.

Demand and supply for a good.

Examine a simple mathematical model to demonstrate the market demand and supply for a product.

Consumers' demand, D, for the product, q^d, depends on the product's price, p, and the consumers' income, Y:

$$q^d = D(p, Y).$$

Firms' supply, S, of the product, q^s, depends on the products price:

$$q^s = S(p).$$

Define the difference between demand and supply, X, of the product by:

$$X := D(p,Y) - S(p). \qquad (2)$$

The product's price will adjust until demand and supply are equal at the equilibrium price, $\underline{p}$, whence:

$$X = D(\underline{p},Y) - S(\underline{p}) = 0$$

Economists ask the comparative statics questions: How do the equilibrium quantity, $\underline{q}$, given by

$$\underline{q} = D(\underline{p}, Y) = S(\underline{p}), \qquad (3)$$

and equilibrium price $\underline{p}$ change with an increase in consumers' income, Y. The obvious response is that both price and quantity will rise.

Since the model does not determine the value of income, Y, it is referred to as an exogenous variable or shift parameter. The model determines the endogenous variables, which are the equilibrium quantity $\underline{q}$ bought and sold at the equilibrium price $\underline{p}$.

Answers to comparative statistics questions are not readily apparent in more complex (realistic?) economic models with numerous endogenous and exogenous variables; instead, they necessitate mathematical investigation. I will illustrate the mathematical analysis of this demand and supply model to show how complex models are handled.

Taking the total derivative of (2) yields with equilibrium variable $\underline{p}$:

$$dX = D_p\, d\underline{p} + D_y\, dY - S_p\, d\underline{p} = 0,$$

which implies after solving for $d\underline{p}$:

$$d\underline{p} = D_y\, dY / (S_p - D_p), \qquad (4)$$

$$\Rightarrow \qquad d\underline{p} / dY = D_y / (S_p - D_p). \qquad (5)$$

Taking the total derivative of (3) yields:

$$d\underline{q} = D_p\, d\underline{p} + D_Y\, dY \Rightarrow d\underline{q} / dY = D_p\, (d\underline{p} / dY) + D_Y.$$

Substituting from (5),

$$d\underline{q} / dY = D_{\underline{p}}\, D_Y / ((S_p - D_p) + D_Y = (D_{\underline{p}}\, D_Y + S_{\underline{p}}\, D_Y - D_{\underline{p}}\, D_Y) / (S_p - D_p),$$

$$\Rightarrow \qquad d\underline{q} / dY = S_{\underline{p}}\, D_Y / (S_p - D_p) \qquad (6)$$

Auxiliary information is required to ascertain whether relations (5) and (6) are positive. If the demand curve moves higher as income rises, or $D_Y > 0$, the product is considered a normal good. Sp > 0 for an upward-sloping supply curve and Dp < 0 for a downward-sloping demand curve.

These prerequisites demonstrate that (5) dp / dY > 0, the equilibrium price will rise, and (6) dq / dY > 0, the equilibrium quantity will rise in response to an increase in income, Y.

Constrained Optimization: Maximization and Minimization.

Constrained optimization is a technique used in microeconomics to study how businesses and consumers achieve their preferred results. Given their financial constraints, consumers purchase a variety of products to optimize their utility functions. Depending on their production functions (technical capabilities), businesses purchase a variety of inputs and create outputs for sale that optimize their profits.

To demonstrate this method, use Archibald's example of a consumer with a budget B who, in order to maximize their objective (utility) function, f, purchases two goods, x1 and x2, at prices p1 and p2. Assume that more of each good will increase the value of the objective function.

Formally:

(7) $$\max f(x1, x2) \text{ such that } B – p1\, x1 – p2\, x2 = 0.$$

This problem can be summed up using the Lagrange function:

(8) $$L(x1, x2, \lambda) = f(x1, x2) – \lambda\, (p1\, x1 + p2\, x2 - B),$$

where λ is called the Lagrange multiplier.

The first order conditions for a maximum are found by taking the derivatives of the Lagrange function, L, wrt to each of its three variable and setting the result equal to zero:

(9)
$$\partial f / \partial x1 = L_1 = f_1 - \lambda\, p1 = 0,$$
$$\partial f / \partial x2 = L_2 = f_2 - \lambda\, p2 = 0,$$
$$\partial f / \partial \lambda = L_3 = -p1\, x1 - p2\, x2 + B.$$

The endogenous variables are x1, x2, and λ, and the exogenous variables are p1, p2, and B.

Resolving these equations for the endogenous variables produces the solution point ($\underline{x1}$, $\underline{x2}$, $\underline{\lambda}$), with its components represented as functions of the exogenous variables, p1, p2, and B:

(10)
$$\underline{x1} = \underline{x1}(p1, p2, B),$$
$$\underline{X2} = \underline{x2}(p1, p2, B),$$
$$\underline{\lambda} = \underline{\lambda}\, (p1, p2, B).$$

These functions, called demand functions, are homogeneous of degree zero. They are invariant to a proportionate increase in the prices and the budget (income).

Take the partial derivative of (8) with respect to the budget variable B => $\partial L / \partial B = \lambda$. Since the budget constraint holds at the values $\underline{x1}$, $\underline{x2}$, and $\underline{\lambda}$: => $L(\underline{x1}, \underline{x2}, \underline{\lambda}) = f(\underline{x1}, \underline{x2})$. The value of the Lagrange multiplier, $\underline{\lambda}$, can be interpreted as the change in the value of the objective function, f, as the budget changes:

(11) $$\partial f(\underline{x1}, \underline{x2}) / \partial B = \underline{\lambda}.$$

The Lagrange multiplier, $\underline{\lambda}$, is sometimes call the marginal utility of money.

Solving the first two equations of (9) at $(\underline{x1}, \underline{x2}, \underline{\lambda})$ for $\underline{\lambda}$:

(12) $$\underline{\lambda} = f_1 / p1 = f_2 / p2 \quad \Rightarrow \quad -f_1 / f_2 = -p1 / p2.$$

The consumer's ability to trade off $\underline{x2}$ for $\underline{x1}$ at prices p2 and p1 is determined by the slope of the budget restriction (-p1 / p2), which equals the marginal rate of substitution (-f1 / f2) evaluated at $(\underline{x1}, \underline{x2})$.

Why is the ratio (-f1 / f2) called the marginal rate of substitution in consumption? An isoquant, iso-f curve, of the objective function, f, is the locus of all points, (x1, x2), such that f(x1, x2) = y, a specific value. Taking the differential, since y is a constant:

(13) $$dy = f_1\, dx1 + f_2\, dx2 = 0.$$

(14) $$\Rightarrow \quad dx2 / dx1 = -f_1 / f_2,$$

which is the slope of the iso-f curve.

With the assumption that the objective function, f, is an increasing function of both x1 and x2 (f_1 and f_2 positive), then (14) is negative and the iso-f curve has a negative slope.

At the solution point, $(\underline{x1}, \underline{x2}, \underline{\lambda})$, the budget line, $p1\,\underline{x1} + p2\,\underline{x2} = B$, is tangent to the iso-f curve, $f(x1, x2) = \underline{y} = f(\underline{x1}, \underline{x2})$.

Multiply the first equation of (9) by $\underline{x1}$ and the second equation by $\underline{x2}$ and add:

(15) $$f_1\, \underline{x1} + f_2\, \underline{x2} = \underline{\lambda}\, (p1\, \underline{x1} + p2\, \underline{x2}) = \underline{\lambda}\, B.$$

Thus, (16) $$\underline{\lambda} = (f_1\, \underline{x1} + f_2\, \underline{x2}) / B.$$

Combine (12), and (16), and (11):

$$\underline{\lambda} = f_1 / p1 = f_2 / p2 = (f_1\, \underline{x1} + f_2\, \underline{x2}) / B = \partial f(\underline{x1}, \underline{x2}) / \partial B$$

The objective function's rate of change with regard to the budget, B, is constant across all margins at the same time.

Check that $\underline{x1}$, $\underline{x2}$, and $\underline{\lambda}$ maximize the objective function subject to the budget constraint.

Take the total differential of the first order equations to obtain the second order conditions:

(16) $$\begin{aligned} f_{11}\, dx1 + f_{12}\, dx2 - p1\, d\lambda &= \lambda\, dp1, \\ f_{12}\, dx1 + f_{22}\, dx2 - p2\, d\lambda &= \lambda\, dp2, \\ -p1\, dx1 - p2\, dx2 &= x1\, dp1 + x2\, dp2 - dB. \end{aligned}$$

Construct the bordered Hessian matrix:

$$(17) \qquad H = \begin{vmatrix} f_{11} & f_{12} & -p1 \\ f_{12} & f_{22} & -p2 \\ -p1 & -p2 & 0 \end{vmatrix}.$$

When the second order partial derivatives, f_{11}, f_{12}, and f_{22}, are evaluated at the solution values, $\underline{x1}$ and $\underline{x2}$, the determinant of H, $|H|$, must be positive, for the objective function, f, to be at a maximum subject to the budget constraint. Write $\underline{H}$ = the matrix, H, evaluated at $(\underline{x1}, \underline{x2})$.

What level of understanding of the function, f, is required to obtain $|\underline{H}| > 0$? The function, f, is an increasing function of both x1 and x2, $f_1 > 0$ and $f_2 > 0$, => marginal utility is increasing. The assumption that marginal utility is diminishing implies $f_{11} < 0$ and $f_{22} < 0$. From the first-order conditions at $(\underline{x1}, \underline{x2}, \underline{\lambda})$, $p1 = f_1 / \underline{\lambda}$, and $p2 = f_2 / \underline{\lambda}$:

$$(18) \qquad |\underline{H}| = \begin{vmatrix} f_{11} & f_{12} & -f_1/\underline{\lambda} \\ f_{12} & f_{22} & -f_2/\underline{\lambda} \\ -f_1/\underline{\lambda} & -f_2/\underline{\lambda} & 0 \end{vmatrix} = (1/\lambda)^2 \begin{vmatrix} f_{11} & f_{12} & -f_1 \\ f_{12} & f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{vmatrix} = (1/\lambda)^2 \; |\underline{F}|.$$

Factoring $(1/\lambda)$ from the last row and column of $|\underline{H}|$:

$$|\underline{H}| = (1/\lambda)^2 \; |\underline{F}|$$

$$(19) \qquad |\underline{F}| = -\{ (f_2)^2 f_{11} - 2 f_1 f_2 f_{12} + (f_1)^2 f_{22} \}.$$

The expression in the brackets, { }, must be negative at the point $(\underline{x1}, \underline{x2})$, to ensure that $|\underline{H}| > 0$.

Using (14) for the slope of an iso-f curve, write:

$$(20) \qquad dx2 / dx1 = r(x1, x2) = -f_1 / f_2.$$

Take the total derivative of (20):

$$(21) \qquad dr = (\partial r / \partial x1)\, dx1 + (\partial r / \partial x2)\, dx2, \text{ or}$$

$$(22) \qquad dr / dx1 = \partial r / \partial x1 + (\partial r / \partial x2)\, dx2 / dx1.$$

But:

$$(23) \qquad \partial r / \partial x1 = -\partial(f_1 / f_2) / \partial x1 = -[f_2 f_{11} - f_1 f_{12}] / (f_2)^2,$$

$$(24) \qquad \partial r / \partial x2 = -\partial(f_1 / f_2) / \partial x2 = -[f_2 f_{12} - f_1 f_{22}] / (f_2)^2.$$

Substituting (20), (23), and (24) into (22):

$$(25) \qquad dr / dx1 = -[f_2 f_{11} - f_1 f_{12}] / (f_2)^2 - [f_2 f_{12} - f_1 f_{22}] / (f_2)^2 \, [-f_1 / f_2].$$

$$(26) \qquad dr / dx1 = -(1 / f_2)^3 \{ (f_2)^2 f_{11} - 2 f_1 f_2 f_{12} + (f_1)^2 f_{22} \}.$$

Convexity of the iso-f curves requires that $dr / dx1 > 0$ => that the expression in the brackets, { }, of (26) is negative => the same expression in (19) is negative.

Therefore, the determinant of the border Hessian matrix, $|\underline{H}|$, is positive, and f takes its maximum value, subject to the budget constraint, at the point ($\underline{x1}$, $\underline{x2}$).

If the iso-f curves are convex, the solution to the first order conditions yields a maximum for the objective function subject to the budget constraint.

What level of understanding of the objective function, f, is required to obtain predictable outcomes, such as the variation in demand for a good resulting from a change in an exogenous variable, p1, p2, or B?

Write the column vectors:

$$d\underline{x} = (d\underline{x1}, d\underline{x2}, d\underline{\lambda})' \text{ and}$$

(27) $$d\underline{p} = (\underline{\lambda}\, dp1, \underline{\lambda}\, dp2, \underline{x1}\, dp1 + \underline{x2}\, dp2 - dB)'.$$

The system of equations (16), evaluated at ($\underline{x1}$, $\underline{x2}$, $\underline{\lambda}$) can be written as the matrix equation:

(28) $\underline{H} * d\underline{x} = d\underline{p}$ where * represents matrix multiplication.

Let $\underline{H\mathbf{i}}$ be the matrix when the ith column of the matrix $\underline{H}$ is replaced by the vector $d\underline{p}$. Then, according to Cramer's rule:

(29) $$d\underline{x\mathbf{i}} = |\underline{H\mathbf{i}}| \,/\, |\underline{H}| = (1/\lambda)^2\ |\underline{Fi}| \,/\, [(1/\lambda)^2\ |\underline{F}|] = |\underline{Fi}| \,/\, |\underline{F}|.$$

Can comparative static results be computed from (29) given our understanding of the qualitative properties of the objective function, f?

For example, what are the signs of $d\underline{x1} / dp1$, $d\underline{x1} / dM$, and $d\underline{x1}/ dp2$?

Set dp2 = dB = 0 in (27):

$$d\underline{p} = (\underline{\lambda}\, dp1, 0, \underline{x1}\, dp1)'.$$

Construct $[\underline{F1}|$:

$$|\underline{F1}| = \begin{vmatrix} \underline{\lambda}\, dp1 & f_{12} & -f_1 \\ 0 & f_{22} & -f_2 \\ \underline{x1}\, dp1 & -f_2 & 0 \end{vmatrix}$$

Then: (30) $$d\underline{x1} / dp1 = [- \underline{\lambda}\, f_2 f_2 + \underline{x1}\, (- f_{12} f_2 + f_{22} f_1)] \,/\, |\underline{F}|.$$

Similarly, for $d\underline{x1}$ / dB with dp = (0, 0, -dB):

Construct $[\underline{F1}|$:

$$|\underline{F1}| = \begin{vmatrix} 0 & f_{12} & -f_1 \\ 0 & f_{22} & -f_2 \\ -dB & -f_2 & 0 \end{vmatrix}$$

(31) $d\underline{x1} / dB = [- (- f_{12} f_2 + f_{22} f_1)] / |\underline{F}|$.

Substituting (31) into (30):

(32) $d\underline{x1} / dp1 = - \underline{\lambda} f_2 f_2 / |\underline{F}| - \underline{x1} (d\underline{x1} / dB)$.

Equation (32) is called the Slutsky equation for $d\underline{x1} / dp1$.

The term, $(d\underline{x1} / dp1)_{comp} = - \underline{\lambda} f_2 f_2 / |\underline{F}| < 0$, is the compensated substitution along the iso-f curve r($\underline{x1}$, $\underline{x2}$) whereby the consumer's income is adjusted to compensate for the price variation.

The term, $(d\underline{x1} / dB)$ < or > 0, is the change in demand for good $\underline{x1}$ as the budget (income) varies.

The Slutsky equation can be written:

(33) $d\underline{x1} / dp1 = (d\underline{x1} / dp1)_{comp} – \underline{x1} (d\underline{x1} / dB)$.

To compute $d\underline{x1} / dp2$, set $dp1 = dB = 0$ in (27):

$d\underline{p} = (0, \underline{\lambda}\, dp2, \underline{x2}\, dp2)'$.

Construct $|\underline{F1}|$:

$$|\underline{F1}| = \begin{vmatrix} 0 & f_{12} & -f_1 \\ \underline{\lambda}\, dp2 & f_{22} & -f_2 \\ \underline{x2}\, dp2 & -f_2 & 0 \end{vmatrix}$$

Then: (34) $d\underline{x1} / dp2 = = [\underline{\lambda} f_1 f_2 + \underline{x2} (- f_{12} f_2 + f_{22} f_1)] / |\underline{F}|$.

Substituting (31) into (34):

(35) $d\underline{x1} / dp2 = \underline{\lambda} f_1 f_2 / |\underline{F}| - \underline{x2} (d\underline{x1} / dB)$.

The new term, $(d\underline{x1} / dp2)_{comp} = \underline{\lambda} f_1 f_2 / |\underline{F}| > 0$, is the compensated substitution effect along the iso-f curve r($\underline{x1}$, $\underline{x2}$).

The Slutsky equation is written:

(36) $d\underline{x1} / dp2 = (d\underline{x1} / dp2)_{comp} – \underline{x2} (d\underline{x1} / dB)$.

By symmetry:

(37) $d\underline{x2} / dp2 = (d\underline{x2} / dp2)_{comp} – \underline{x2} (d\underline{x2} / dB)$.

(38) $d\underline{x2} / dp1 = (d\underline{x2} / dp1)_{comp} – \underline{x1} (d\underline{x2} / dB)$.

In general, the Slutsky equation for each individual good and price says: the total effect equals the compensated substitution effect plus the income effect. The good, $\underline{xi}$, is normal if $(d\underline{xi} / dpi) > 0$; Giffin if $(d\underline{xi} / dpi) < 0$; it is superior if $(d\underline{xi} / dB) > 0$, inferior $(d\underline{xi} / dB) < 0$.

What about $d\underline{\lambda} / dB$, $d\underline{\lambda} / dp1$, and $d\underline{\lambda} / dp2$?

Set dp1 = dp2 = 0: d$\underline{p}$ = (0, 0, - dB)'.

Construct |$\underline{F3}$|:

$$|\underline{F3}| = \begin{vmatrix} f_{11} & f_{12} & 0 \\ f_{12} & f_{22} & 0 \\ -f_1 & -f_2 & -dB \end{vmatrix}$$

(39) $d\underline{\lambda} / dB = [- (f_{11} f_{22} – f_{12} f_{22})] / |\underline{F}|$.

From (17), let D be the matrix of second order partial derivatives of the objective function, f. The determinant of D evaluated at ($\underline{x1}$, $\underline{x2}$) is $|\underline{D}| = (f_{11} f_{22} – f_{12} f_{12})$. For equation (39) write:

(40) $d\underline{\lambda} / dB = d (\partial f(\underline{x1}, \underline{x2}) / \partial B) / dB = - |\underline{D}| / |\underline{F}| = \underline{\mu}$.

Then, $\underline{\mu}$ is a measure of the rate of decrease in the marginal utility of income (budget). If $\underline{\mu} < 0$, then $|\underline{D}| > 0$. If f($\underline{x1}$, $\underline{x2}$) were an unconstrainted maximum then $f_1 = f_2 = 0$ at ($\underline{x1}$, $\underline{x2}$, $\underline{\lambda} = 0$)

To compute d$\underline{\lambda}$ / dp1, set dp1 = dB = 0: d$\underline{p}$ = ($\underline{\lambda}$ dp1, 0, $\underline{x1}$ dp1)'.

Construct |$\underline{F3}$|:

$$|\underline{F3}| = \begin{vmatrix} f_{11} & f_{12} & \underline{\lambda}\, dp1 \\ f_{12} & f_{22} & 0 \\ -f_1 & -f_2 & \underline{x1}\, dp1 \end{vmatrix}$$

(41) $d\underline{\lambda} / dp1 = [\underline{\lambda} (-f_{12} f_2 + f_{22} f_1) + \underline{x1} (f_{11} f_{22} – f_{12} f_{22})] /|\underline{F}|$.

Substituting (31) and (39) into (41):

(42) $d\underline{\lambda} / dp1 = - \underline{\lambda} (d\underline{x1} / dB) – \underline{x1} (d\underline{\lambda} / dB)$

By symmetry:

(42) $d\underline{\lambda} / dp2 = - \underline{\lambda} (d\underline{x2} / dB) – \underline{x2} (d\underline{\lambda} / dB)$

Goods $\underline{x1}$ and $\underline{x2}$ are substitutes:

(43) $p1 (d\underline{x1} / dp1)_{comp} + p2 (d\underline{x1} / dp2)_{comp} = - (f_1 / \underline{\lambda}) \underline{\lambda} f_2 f_2 + (f_2 / \underline{\lambda}) \underline{\lambda} f_1 f_2 = 0$.

Since p1, and p2 are positive, and $(d\underline{x1} / dp1)_{comp}$ is negative, $d\underline{x1} / dp2)_{comp}$ is positive => $\underline{x1}$ and $\underline{x2}$ are substitutes.

Archibald's fundamental Theory of the Household model presents insights applicable in broader frameworks. His model explains how comparative statics reveals the change in the value of endogenous variables from one equilibrium state to another with respect to a change in value of an exogenous variable. Changing the bundle of goods from two items, **x** = (x1, x2), to a greater number of items, **x** = (x1, ..., xn), does not change the analysis. However, for two goods to be complements, $(d\underline{xi} / dpj)_{comp} < 0$, the model needs at least three goods, as Archibald's course notes indicate.

From 1975 to 1977, I taught graduate level Mathematical Economics at Carleton University. My course text was Michael D. Intrilligator's book, *Mathematical Optimization and Economic Theory*. The analysis of a household consuming a general bundle goods, x = (x1, ..., xn), necessitates the use of vectors, matrices and determinants of appropriate dimensions. When the first order conditions are evaluated at their solution vector **x** = (x1, ..., xn), the second order conditions are a bordered Hessian matrix of dimension (n+1) x (n+1). Comparative static results can be calculated more easily when partial derivatives are used rather than total differentials with more than two goods. However, Archibald's method makes the principles easier to understand.

The standard claims of household demand—homogeneity, Engels aggregation, symmetry, Cournot aggregation, and integrability of an iso-f surface to a well-defined utility function—are examined by both Archibald and Intrilligator using a broad bundle of goods.

A department store, such as the T. Eaton Company, where I worked in 1962, sold a wide range of products to households. For instance, the men's wear area offered a variety of suits, shirts, and ties. When buying a suit, a consumer would also buy a few shirts and ties. A consumer who purchased one shirt looked at the other shirts as alternatives. Suits, shirts, and ties were complementary items when purchased together; when purchased separately, they were a group of substitutes.

Shopping at Eaton's was an experience many Canadians remember. Their quality merchandise was backed by their "Satisfaction Guaranteed or Money Refunded" slogan. A month before Christmas, children's toys would appear, and just as magically, disappear after New Years. Customers postponed purchases in anticipation of the twice-yearly "Trans-Canada Sale." The less costly sale items served as a satisfactory substitute for the normal items for consumers on a limited budget.

The Eaton's mail-order catalogue provided a viable alternative to shopping at a city store for those who lived in rural locations. With mail-in forms, people could order toys, clothes, household appliances and furniture, farm equipment, and even whole prefabricated homes shipped by train. Prior to the widespread use of television, the catalogue served as a marketing instrument. Customers could browse the catalogue, compare prices with other retailers, and decide if things were within their budget before purchasing them at the nearest Eaton's store. Out-of-stock items could be ordered in-store with personalized customer service.

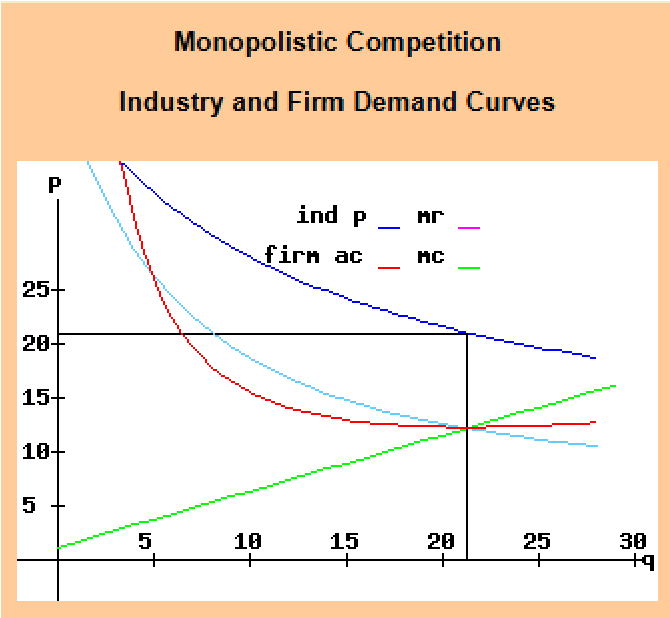
More Microeconomic Models and Concepts.

Fall 1972: Economics 500. Markets and firms: perfect competition, monopolistic competition, oligopoly, monopoly, production functions, technological change. Revealed preference theory, income distribution, von-Neumann-Morgenstern utility, welfare economics, theory of the second best.

Monopolistic Competition.

URL: https://www.egwald.ca/economics/monopolistic1.php

The many firms in a monopolistically competitive industry produce differentiated yet similar products. New firms can easily enter the industry. Textbooks often give retail trade or the hotel industry as examples.



A monopolistically competitive firm's own demand curve is highly elastic, permitting it to vary its price within a narrow range of prices. The other firms' products are either very close substitutes or, a large number of other firms' products are substitutes (not necessarily very close substitutes).

The diagram shows the industry and firm's demand and marginal revenue curves. The slopes of the firm's demand and marginal revenue curves are greatly exaggerated. In reality, they are almost coincident horizontal lines.

The firm's own demand schedule intersects the "industry's (average) demand schedule" at the firm's "equilibrium level of output and price," given the output levels of the other firms in the industry. These concepts are explained below.

Almost every day I get "flyers" in the mail informing me about the products of a new or existing restaurant, clothing store, or laundry, and their terms of trade (relative prices). Monopolistically competitive firms advertise (unlike perfectly competitive firms). Advertising seems necessary for a firm to enter and to remain in the industry.

Retail firms pay taxes on the value of products sold (in Canada - G.S.T. and provincial sales taxes). The effect of varying these taxes on firms will be built into the model of a monopolistic industry.

Suppose there are n firms in the industry. If q is the output of product and v is the amount of advertising of a typical firm, then the price, p, of its product is:

$$p(q,v) = A / (n * q + B) + ad(v)$$

A and B are positive constants. Since we are modelling the typical firm, in equilibrium each of the n firms will sell the same amount of product. Each firm's advertising function ad(v) shifts the demand curve. It is given by:

$$ad(v) = c1*v - c2*v^2$$

where c1 and c2 are positive constants. The negative sign ensures diminishing returns to advertising. Without advertising, a firm's sales prospects are diminished.

Including the term, **B**, as a positive constant in the "industry's (average) demand schedule" ensures a "finite price," even when n = 0, or q = 0.

The cost function for the typical firm is: $c(q,v) = a*q^2 + b*q + c + s*v$

where a, b, c, and s are positive constants. Fixed costs are (c + s*v) because they do not vary with q.

The average cost function will be U-shaped: $ac(q,v) = a*q + b + (c + s*v) / q$

With an ad valorem tax at rate t (sales tax rate = t / (1-t)), the revenue to the firm is:

r(q,v) = q * p(q,v) * (1 - t)

At given levels of output, q, and advertising, v, the after-tax profit of the typical firm is given by:

prof(q,v) = r(q,v) - c(q,v)

If the firm sets q and v to maximize its profit, the following first order marginal conditions must obtain:

$prof_q = r_q - c_q = 0$; $prof_v = r_v - c_v = 0$

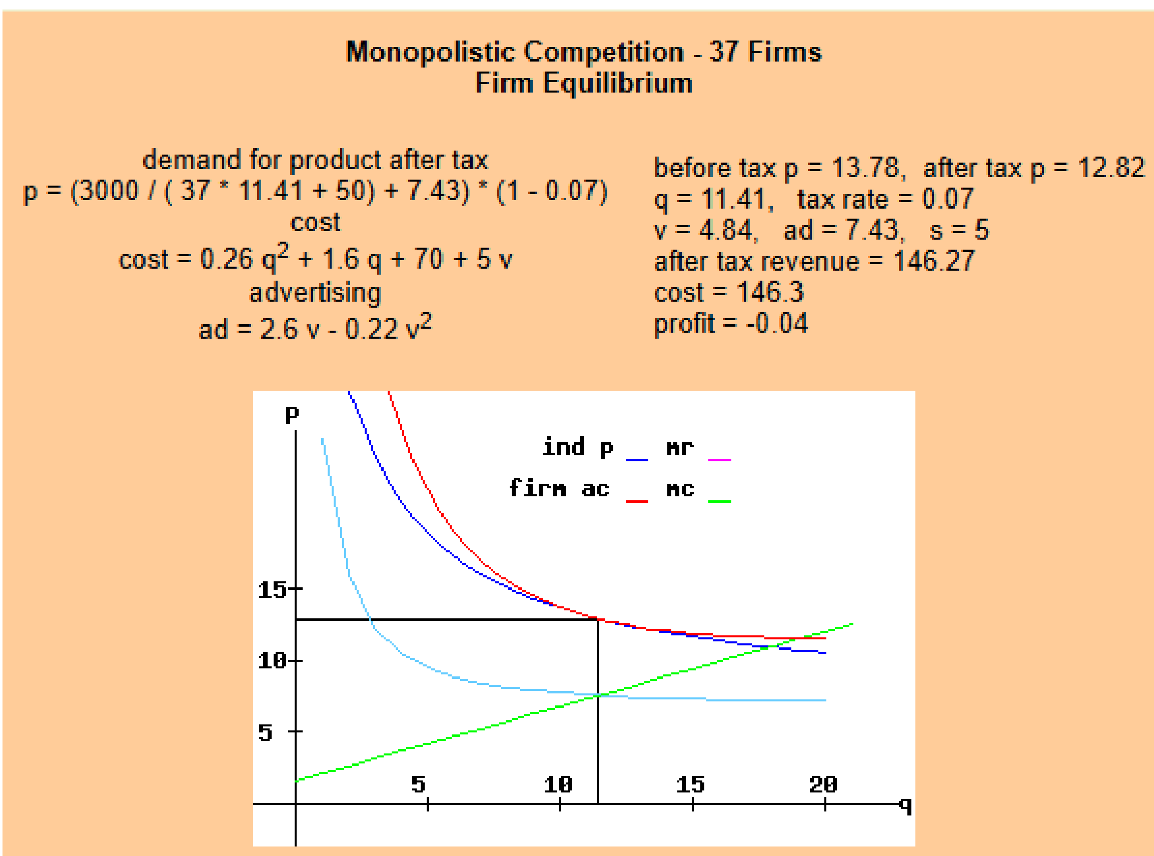
where the subscripts represent the partial derivative of the function with respect to the specified variable.

These first order conditions plus their associated Jacobian used for each set of parameters of the model to solve for the optimal value of q and v.

Suppose, first, that all firms in the industry behave (in concert) as the typical firm. Each firm will attempt to satisfy the above first order conditions by selecting appropriate levels of q and v.

Firms will enter (or exit) the industry until the after-tax price equals the average cost at the equilibrium level of q, the typical firm's output. At this equilibrium q and equilibrium number of firms n, the average cost curve will be tangent to the average revenue curve, p(q,v), and profits ~= 0.

Such an equilibrium is shown below.



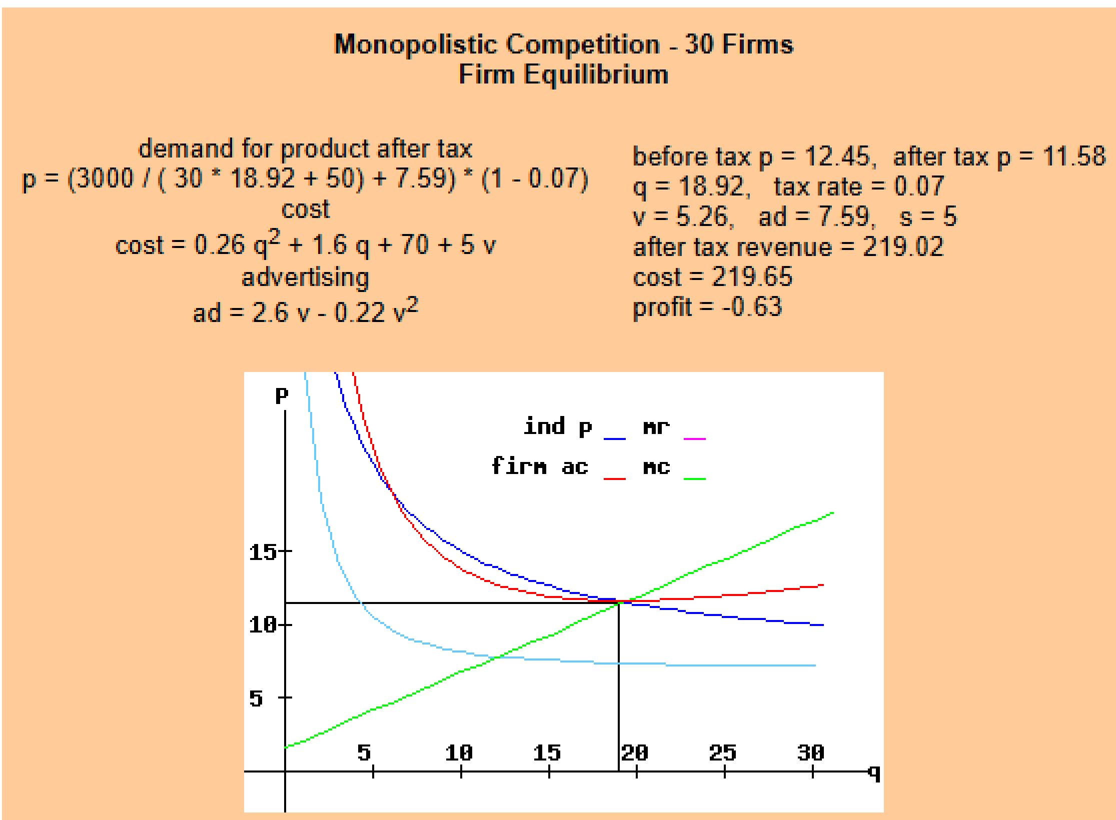
However, the typical firm's **own** demand schedule is given by:

p(q,x,v) = A / ((n-1) * q + x + B) + ad(v)

where the other (n-1) firms are at their equilibrium level of output q, while the typical firm varies its own output x.

In the diagram above, we can see that the firm's **own** average revenue curve, p(q,x,v), intersects the ac(q,v) and p(q,v) curves at their point of tangency. Here the firm's **own** marginal revenue, mr(q,x,v,) curve lies above its marginal cost curve, mc(x,v). Thus, it may try to increase its sales by cutting its price slightly. If the other firms follow suit, this price cutting will lead to an actual loss in revenue and an increase in the firm's costs with profit turning negative. Some firms might exit, or firms might increase prices to their former level, restoring the zero-profit condition.

If the industry's firms are aggressive enough, we might expect, **provided it is feasible,** another "equilibrium" to obtain: with sales at that level of ouput where each firm's **own** average revenue curve, p(q,x,v), is tangent to its **own** average cost curve, ac(x,v), and where its **own** marginal revenue, mr(q,x,v), equals marginal cost, mc(x,v). This is the equilibrium below.



The industry's price as a function of n, q, and v is: **p(q,v) = A / (n * q + B) + ad(v),**
where advertising as a function of v is: **ad(v) = c1*v - c2*v^2**
A firm's costs as a function of q and v is: **c(q,v) = a*q^2 + b*q + c + s*v.**
A firm's revenue as a function of q and v is: **r(q,v) = q * p(q,v) * (1 - t).**
A firm's **own** demand schedule as a function of x and v is: **p(q,x,v) = A / ((n-1) * q + x + B) + ad(v).**

A firm's **own** costs as a function of x and v is: **$c(x,v) = a*x^2 + b*x + c + s*v$.**
A firm's **own** revenue as a function of x and v is: **$\underline{r}(q,x,v) = x * \underline{p}(q,x,v) * (1 - t)$.**

Oligopoly.

URL: https://www.egwald.ca/economics/econpage.php

Oligopoly is a market structure with few firms and barriers to entry, according to economists. Oligopolists understand how interdependent they are. Economists have employed the concept of conjectural variation to explain how oligopolists respond to the actions of their rivals in their common markets. The management of a company speculates (conjectures) on how competitors may respond by varying their prices and outputs when they consider changing either. Although this generalization is logical, it is challenging to translate it into a mathematical model that can be tested empirically.

When I worked at Canada Packers in 1970, the company had four competitors of similar size, but its sales were double that of its two biggest rivals combined. In its research lab, new production technologies were created and implemented. Its profits and leading position in the Canadian food processing sector were preserved by consistent investment in new plants, facility expansions and modernization, domestic and international acquisitions, product diversification (such as medicines), and the creation of trading companies in England and Germany.

The management of Canada Packers (CP) was aware of how its choices might impact its primary rivals strategically. One of its vice-presidents, Dr. Clarke, had been the Company President W. F. McLean's chemistry professor at Standford University. My colleagues in operations research were knowledgeable about game theory and updated the management on new advancements. CP forestalled its competitor McCain Feed's takeover bid by repurchasing 3.5% of its issued shares.

CP was able to function with noteworthy economies of scale because of its large investment in industrial infrastructure, which significantly hindered the entry of new enterprises. CP's cheap prices discouraged new entrants in industry sectors where it was possible for new enterprises to enter, such as the production of margarine. Notwithstanding the losses, this tactic was used.

As previously stated, Canada Packers did not collude with other meat packers; yet, in the 1980s, its rivals were found guilty of price-fixing. Their goal was to better compete with CP by illegally synchronizing their marketing activities.

In 1970, CP and other meat packers were deterred by the Canadian government from integrating downstream and selling in the retail sector. Through the Department of Consumer and Corporate Affairs' Combines Investigation Branch, the Canadian government regulated anti-competitive actions such as price-fixing, monopolies, and mergers that unnecessarily reduced competition.

When I was a lecturer at Carleton University in 1978, I examined the advantages and disadvantages of vertical integration by a dominant firm: Section III, Vertical Integration: Efficiency Versus Restricted Output, of my Carleton Economic Papers No, 78-09, "Government Firm Regulation of a Vertically Integrated Industry" (https://www.egwald.ca/wiens/elmerwiensgovregvertintindustry.pdf) I demonstrate that attempts by the government to "limit the natural evolution of organizational forms, in the interests of 'increased competition,' may impair efficiency and may in fact decrease output."

These days, 2025, Loblaw Companies / President's Choice production and marketing strategy includes backward integration (private label production) and forward integration (retail, pharmacy, and financial services), making it a textbook example of vertical integration in Canadian retail.

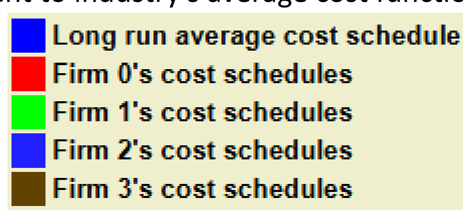
In the supermarket sector, its rivals are less unified. Vertical integration was demonstrated by Canada Safeway, a significant grocery store chain. Prior to being acquired by Sobeys, Canada Safeway owned and operated some of its supply chain, such as distribution centers and private-label production for specific food items. It managed sourcing, logistics, and retailing—features symptomatic of vertical integration.

In Western Canada, Save-On-Foods, a division of the Overwaitea Food Group, has a more regional focus. Although it emphasises local sourcing and strong supplier connections, evidence of vertical integration is lacking. Most of its supply chain is dependent on outside logistics and suppliers. It doesn't own any farms or industrial facilities. It partners with a wide range of producers and manufacturers—many of them Canadian—to create its private-label Western Family products according to its specifications and quality standards.

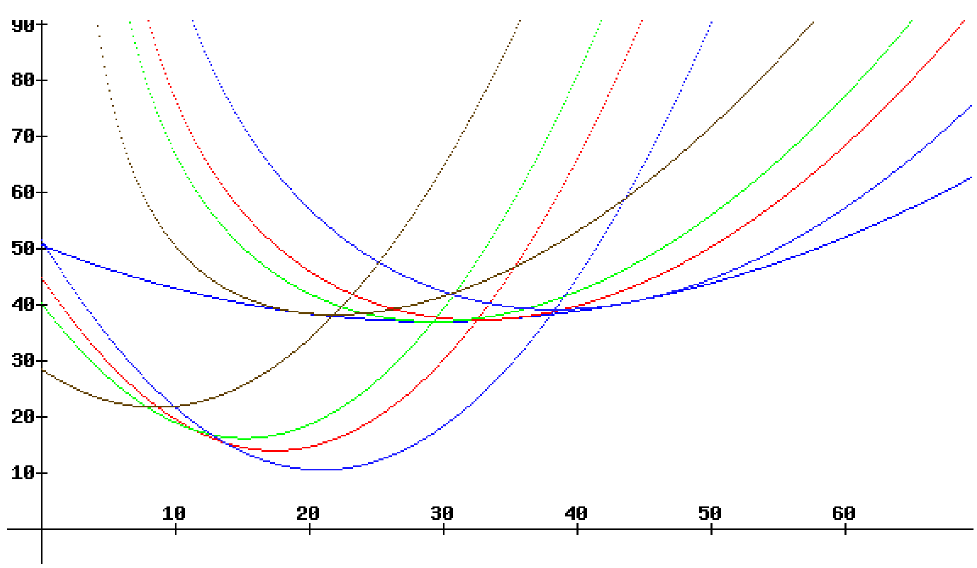
Diagrams of Firms in an Oligopoly.

Imagine a market where four companies in a given industry are manufacturing nearly similar goods. Their short-run (capital fixed) average cost schedules, which are tangent to the industry's long-run (capital variable) average cost schedule, reflect the firm's size differences. The product's short-term demand is reflected in the demand schedule.

Demand function: p = 83.6871 -0.4692 * q + 0.0005017 * q^2
Industry's Average Cost Functions: Long run quadratic: 50.2233 -0.9174 * q + 0.0156 * q^2
Firms' Average Cost Functions: Short run quadratic**:** Tangent to industry's average cost function



Firm 0 : 459.817 / q + 44.531 -1.752 * q + 0.0334 * q^2
Firm 1 : 388.862 / q + 39.953 -1.568 * q + 0.0343 * q^2
Firm 2 : 643.344 / q + 51.162 -1.939 * q + 0.0308 * q^2
Firm 3 : 267.511 / q + 28.268 -0.786 * q + 0.0305 * q^2

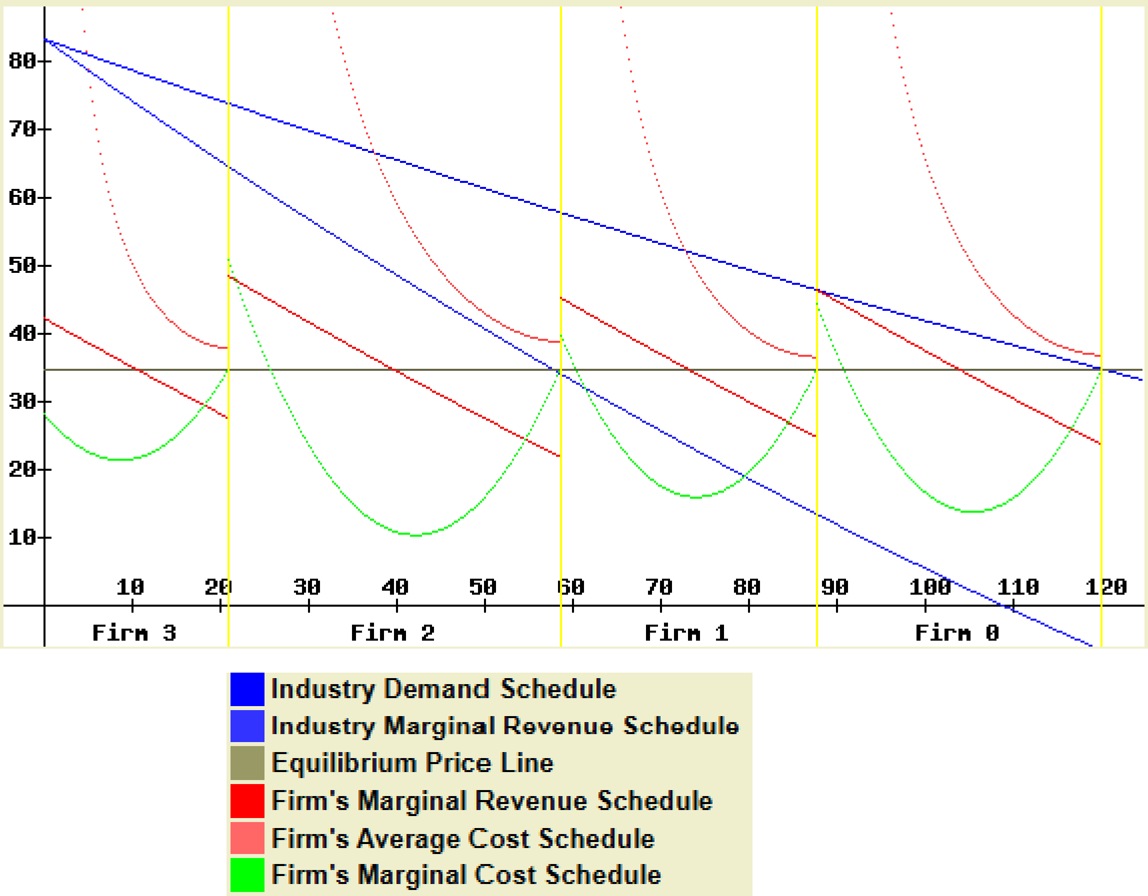


Firms Are Price Takers.

Firms produce output at a level where price equals marginal cost.

firm	output quantity	marginal cost	marginal revenue	average cost	total cost	revenue	profit
0	32.03	35.01	23.8	37.01	1185.35	1121.18	-64.16
1	28.8	35.01	24.93	36.75	1058.48	1008.21	-50.27
2	37.31	35.01	21.95	38.91	1451.8	1306.07	-145.73
3	20.72	35.01	27.76	38	787.31	725.38	-61.93

Industry: price : 35.01 **output :** 118.85 **MR :** -6.58 **elasticity:** -0.84 **profits:** -322.09



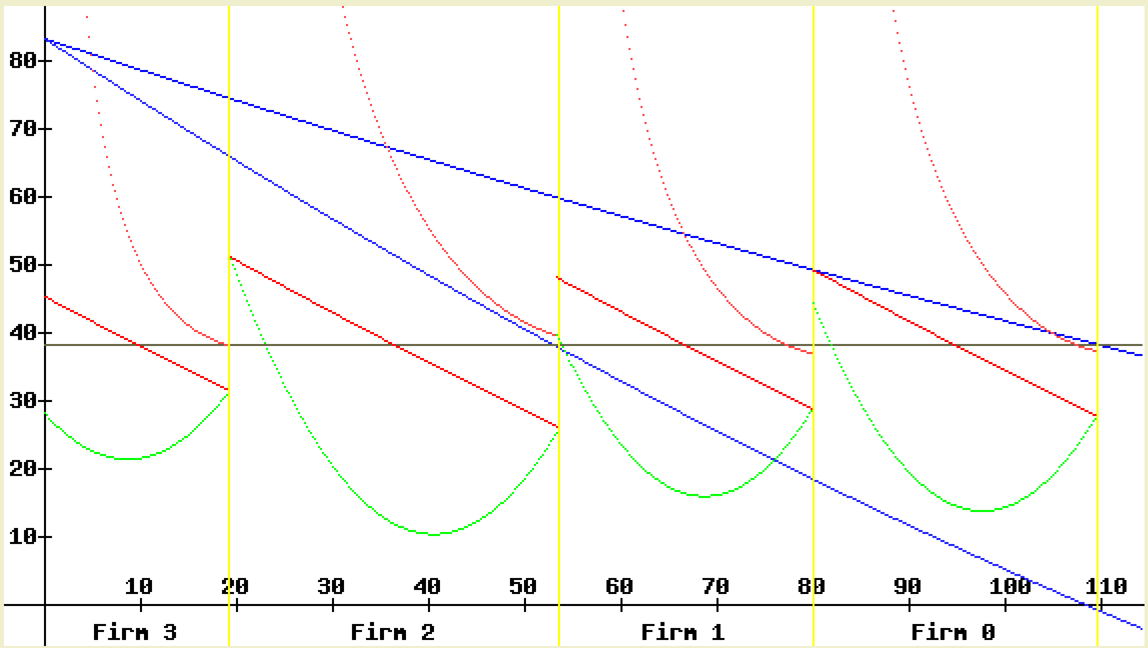
All firms are operating at a loss, negative profits. This is an untenable situation. A firm must leave the industry or the firms must somehow coordinate their output levels.

Firms Use Their Own Marginal Revenue Schedule.

Each firm takes the other firms' total output as given, and produces output at a level where its marginal cost equals its marginal revenue. This is a Nash equilibrium because a firm cannot increase its profits independently. No collusion among firms is necessary for this outcome.

firm	output quantity	marginal cost	marginal revenue	average cost	total cost	revenue	profit
0	29.35	27.95	27.95	37.53	1101.57	1130.57	29.01
1	26.44	29	29	37.19	983.29	1018.42	35.14
2	34.08	26.25	26.25	39.71	1353.45	1312.55	-40.9
3	19.1	31.64	31.64	38.4	733.28	735.51	2.23

Industry: price : 38.51 **output :** 108.97 **MR :** -0.7 **elasticity:** -0.98 **profits:** 25.48



Three firms are operating with profits, but firm 2 with the largest fixed cost operates at a loss.

Firms Collude to Restrict Output: Cartel.

What level of output would maximize the firms' profits? Suppose firms attempt to equalize their marginal costs and operate as a cartel.

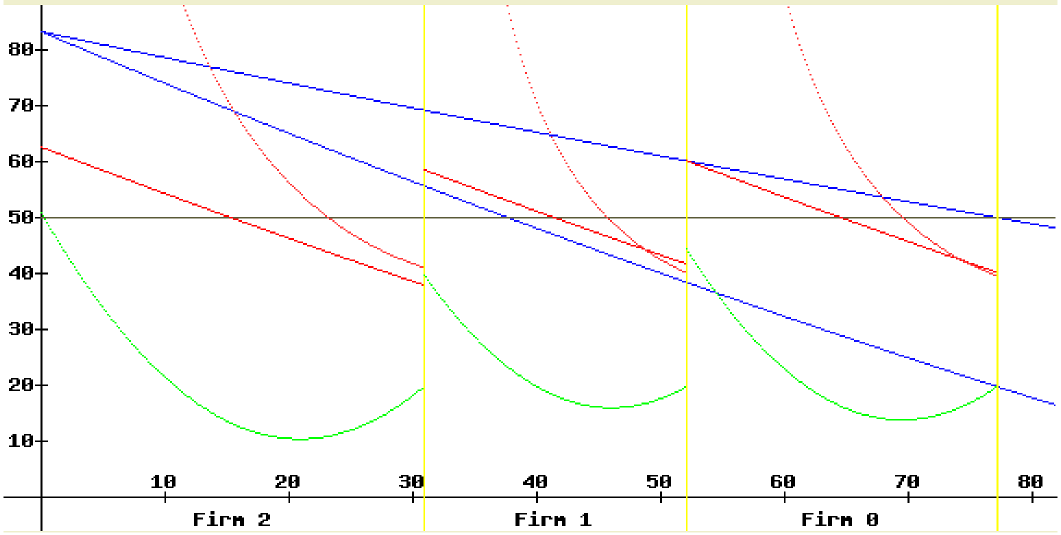
firm	output quantity	marginal cost	marginal revenue	average cost	total cost	revenue	profit
0	23.57	17.57	40.22	41.29	973.18	1164.14	190.96
1	18.9	17.45	42.03	43.14	815.53	933.55	118.02
2	29.93	17.82	37.74	42.2	1263.18	1478.33	215.15
3	7.53	21.62	46.46	59.62	448.78	371.77	-77.02

Industry: price : 49.39 **output :** 79.94 **MR :** 18.29 **elasticity:** -1.59 **profits:** 447.11

Firm 3 with the smallest fixed cost has the highest marginal cost and operates at a loss. The other firms obtain a profit. Firm 3 leaves the industry, perhaps with a monetary inducement from other firms.

firm	output quantity	marginal cost	marginal revenue	average cost	total cost	revenue	profit
0	25.24	19.88	40.39	39.79	1004.39	1268.99	264.6
1	21.32	19.88	41.93	40.35	860.45	1071.91	211.46
2	31.1	19.88	38.1	41.32	1285.22	1563.63	278.41

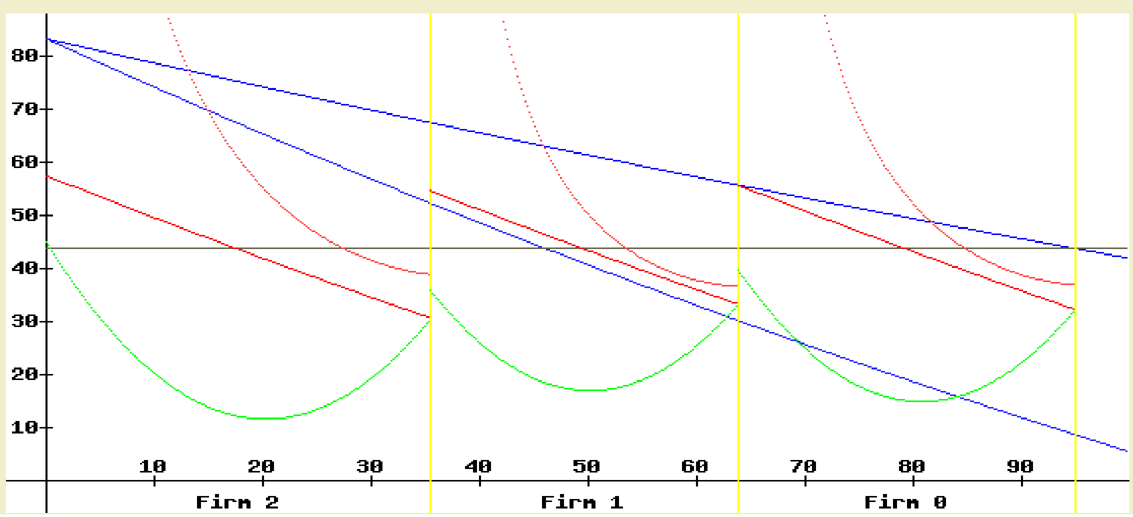
Industry: price : 50.27 **output :** 77.67 **MR :** 19.88 **elasticity:** -1.65 **profits:** 754.46



Industry profits have increased by 230.33 to 754.46. Might these firms revert to the Nash equilibrium?

firm	output quantity	marginal cost	marginal revenue	average cost	total cost	revenue	profit
0	30.81	32.43	32.43	37.11	1143.26	1354.71	211.45
1	28.1	33.44	33.44	36.81	1034.56	1235.77	201.2
2	35.2	30.78	30.78	39.07	1375.52	1547.96	172.44

Industry: price : 43.97 **output :** 94.11 **MR :** 8.7 **elasticity:** -1.25 **profits:** 585.1



Each firm's profits are less in the 3-firm Nash equilibrium than in the 3-firm cartel. Firm 1 with the lowest fixed cost and the highest own marginal revenue has the greatest incentive to cheat on the cartel.

Spring 1972: Economics 501. Theory of the allocation of time, minimum wages in B.C., Le Chatelier principle, duality between production and cost functions, general equilibrium, temporary and full equilibrium, market stability, Arrow's possibility theorem, externalities, the size distribution of firms, alternative theories of the firm, and firm growth models

Macroeconomics.

1972-73. The first-year graduate course in macroeconomics was taught in two sections, one per semester. Dr. Robert Evans taught the fall semester course, Macroeconomics. Dr. Keizo Nagatani taught the spring semester course, Economic Fluctuations and Growth.

Economics 502.

On the first day of classes, Dr. Evans issued a fifteen-page course syllabus, outlining students' responsibilities and expectations. The first instruction was "You should have read Keynes' *General Theory* by now – if not do so." There were roughly 135 recommended articles and book chapters to read over the nine sections of the course. There was no textbook for the course, although W. H. Branson's *Macroeconomic Theory and Policy* graduate level textbook was published that year.

The presentation was more formal than intermediate level macroeconomics, "using linearity as an approximation of a totally differential general function and solving by explicit matrix methods to get comparative statics results." In the days before easy access to university and / or personal computers, classroom models were not quantitative.

Models analyzed using comparative statistics were able to anticipate how changes in the values of exogenous variables would affect the endogenous variables. Without calibrating the model equations, however, it is impossible to determine the exact extent of these adjustments in the exogenous variables.

The introduction of the internet allowed for the presentation of quantitative models for public review. In economics jargon, an open macroeconomics model includes exports, imports, and capital (money) flows. My nonlinear quantitative model of the open Canadian economy appears at the URL: https://www.egwald.ca/macroeconomics/basicislmexternal.php. The parameters of the model's macroeconomics functions reflect the average values of the macroeconomics aggregates of the Canadian economy during the years of 2001 - 2003.

Section I: IS-LM Model.

The Basic GNP Identity:

$$Y \equiv C + I + G + (X - M) \equiv GNP = C + S + T + R. \quad (1)$$

The term Y = Gross National Income (GNI) = income received for productive activity by residents. The middle expression measures GNP by expenditures on final product, and the right-hand side measures GNP by the way income earned in production is disposed.

C = consumer expenditure, I = gross private domestic investment, G = total government purchases of goods and services by all levels of government, X = exports, M = imports, (X – M) net exports plus net factor receipts from abroad, S= total saving by consumers and by businesses–like retained earnings and depreciation allowances, T = net tax payments, and R = net international transfer receipts.

Writing GDP = value of output produced domestically, then GNP – GDP = net factor payments from foreign countries.

The following identities and equations describing the sectoral balances are from *Open Economy Macroeconomics* by Rudiger Dornbusch.

Sectoral Balances.

From equation (1):

$$Y - C - I - G \equiv X - M: \tag{2}$$
$$(X - M) > 0 \rightarrow \text{spending less than income},$$
$$(X - M) < 0 \rightarrow \text{spending exceeds income}.$$

According to relations (2), issues with the external balance must have a macroeconomic component. Their remedy must involve a way to restore the equilibrium between income and expenditure.

Subtract T, net tax payments, and add R, net international transfer receipts, to the first two expressions of equation of (1):

$$Y + R - T \equiv C + I + (G - T) + (X + R - M) \tag{3}$$

Equation (3) reads that disposable income of domestic residents equals by definition the sum of domestic consumption and investment, the government deficit, and the current account surplus.

Define domestic saving:

$$S \equiv Y + R - T - C. \tag{4}$$

Substitute (4) into (3) and rearrange:

$$(S - I) + (T - G) \equiv (X + R - M) \tag{5}$$

Equation (5) reads that the excess of private sector savings over investment plus the budget surplus equals by definition the current account surplus. A positive current account implies that domestic residents and firms are adding to their net foreign assets.

To conclude, Dornbush states that the three sectoral balances, (3), (4), and (5) are simultaneously obtained by the way the economy determines the general equilibrium of income and price.

Dr Evans introduced the basic, linear IS-LM model of a closed economy (no exports, imports, or capital flows). The IS curve models the product market; the LM curve models the money market.

Product Market IS curve.

Define disposable income available by consumers:

$$Y_d = Y - T. \tag{6}$$

The behavioural relationship describing consumer's consumption:

(7) $C = C_o + C_1 Y_d = C_o + C_1 (Y - T)$.

The behavioural relationship describing firm's investment, where r is the rate of interest:

(8) $I = I_0 + I_1 - I_2 r$.

The institutional relationships describing taxes collected by the government and government expenditures:

(9) $T = T_0 + T_1 Y$; $G = \underline{G}$.

Product market equilibrium: Substituting (7), (8) and (9) into Y = C + I + G, and collecting terms:

(10) $Y = C_o + C_1 Y - C_1 T_0 - C_1 T_1 Y + I_0 + I_1 Y - I_2 r + \underline{G}$.

IS curve: Solving (10) for Y:

(11) $Y = [C_o - C_1 T_0 + I_0 - I_2 r + \underline{G}] / [1 - C_1 + C_1 T_1 - I_1]$.

By inspection of (11), income, Y, is a decreasing linear function of the interest rate, r, with slope $-I_2$ divided by the denominator. The other parameters in (11) will shift the IS curve upward or downward depending on their signed values

Money Market LM Curve.

The money supply:

(12) $M_S = \underline{M}$.

The demand for money:

(13) $M_D = M_0 + M_1 Y - M_2 r$

Where $M_1 Y$ is the transactions demand and $M_2 r$ is the speculative demand for money.

Money market equilibrium:

(14) $M_D = M_0 + M_1 Y - M_2 r = M_S$.

LM Curve: solving (14) for Y:

(15) $Y = [M_S - M_0 + M_2 r] / M_1$.

By inspection of (15), income, Y, is an increasing function of the interest rate, r, with slope, M_2 / M_1. The other parameters in (15), M_0 and M_1, will shift the LM curve upwards or downwards depending on their values.

My nonlinear quantitative model of the closed Canadian economy appears at the URL: https://www.egwald.ca/macroeconomics/basicislm.php. It shows both algebraically and graphically how the IS and LM curves are derived.

Labour Market.

Using diagrams, the instructor described the Classical and Keynesian versions of the labour market. The classical labour market assumes that full employment is guaranteed by flexible wages. The Keynesian labour market recognizes that demand shortages and wage stickiness result in involuntary unemployment.

The labour market was not integrated with the IS-LM model.

The following two paragraphs are from my online Aggregate Demand – Aggregate Supply model of the Canadian economy at the URL: https://www.egwald.ca/macroeconomics/keynesian.php.

The Keynesian IS-LM model focuses on the "demand side" of the economy - the relationship between national income and the aggregate demand for product (goods and services) by consumers, producers, and governments. The IS-LM model ignores the price level of goods and services, the level of employment, the wage rate of workers, and the amount of product output. To include these macroeconomics aggregates in the model, I combined the labour market with the aggregate relationship between employment of workers and the level of product output, providing the aggregate supply equation - a relationship between the price level and the level of output produced by firms. I also modified the equations that describe the money market of the basic IS-LM model to reflect the effect that changes in the price level have on the perceived "real supply" of money, yielding the aggregate demand equation - a relationship between the price level and the demand for output of firms.

The intersection of the aggregate demand and aggregate supply equations will yield the equilibrium level of output, the price level, the wage rate, and the level of employment, along with the rate of interest and the values of all the other macroeconomics variables obtained from the IS-LM model. This aggregate demand-aggregate supply (AD-AS) economics model tries to approximate the relationships among these key macroeconomics aggregates.

Comparative statics analysis of the AD-AS model can be done at the URL: https://www.egwald.ca/macroeconomics/compstatskeynesian.php.

On this dynamic webpage, you can adjust the Classical & Keynesian AD-AS models' parameters and see how the values of the equilibrium macroeconomics variables and aggregates change. By changing the parameters of the labour market and the aggregate production function, you can shift the economy's aggregate supply schedule. Moreover, by changing the parameters of the product and money market, you can shift the parameters of the economy's aggregate demand schedule. The intersection of the aggregate supply and demand schedules determines the economy's equilibrium price level, and national income & product. Feeding these equilibrium values into the equations that underlie their schedules determines the level of employment, the money wage rate, the economy's wage bill, the interest rate, the demand for money, consumer expenditures, investment in capital by firms, and government revenues.

Section II: **Elementary Dynamics.**

Dynamical systems can be modelled by either differential or difference equations. Differential equations model continuous systems where variables change smoothly. Differential equations model discrete systems where variables change at specific point in time.

The next paragraph is from my online analysis of the Trygve Haavelmo Growth Model at the URL: https://www.egwald.ca/nonlineardynamics/haavelmogrowth.php.

The Trygve Haavelmo growth model in *A Study in the Theory of Economic* Evolution provides an example of the different dynamical behaviours arising from equivalent models expressed as either differential or difference equations. While the solution trajectories of the continuous time version of the model converge to its fixed point, the solution trajectories of the discrete time version may exhibit chaotic behaviour.

Section III: Exercises with the Small, Semi-Dynamic Model.

To quote the syllabus: A number of particular problems can be examined by manipulation of the simple IS-LM model, adding variables or lags, playing with the government sector, policy rules, accelerators, inventory models etc.

Section IV: National Accounts.

See Statistics Canada National Economic Accounts URL: https://www150.statcan.gc.ca/n1/pub/13-607-x/2016001/36-eng.htm.

Section V: Consumption in Theory and Estimation.

IS-LM Consumption Function: $C = C_o + C_1 Y_d = C_o + C_1 (Y - T)$.

For the purpose of learning about "permanent income, life cycles, intentions and expectations, and consumption / wealth relations," the instructor recommended that students read about twenty-five articles and book chapters. The learner would presumably have a more profound comprehension of the IS-LM model's consumption function.

Section VI: Investment, Capital, and All That.

IS-LM Investment Function: $I = I_0 + I_1 - I_2 r$.

In order to learn about models of investment behavior and their empirical estimation, the class read roughly twenty papers and book chapters over the course of two weeks.

The marginal efficiency of capital (MEC) proposed by Keynes was one of the subjects examined. MEC, which takes into consideration the potential yield of an investment over its lifetime, is by definition the expected rate of return on an additional unit of capital investment.

The internal rate of return (IRR) is not the same as the marginal efficiency of capital. The discount rate that brings the net present value (NPV) of the cash flows from an investment to zero is known as the internal rate of return, or IRR. It stands for an investment project's anticipated rate of return.

Section VII: Fiscal Policy.

IS-LM Fiscal Policy: $T = T_0 + T_1 Y$; $G = \underline{G}$.

How can the government influence the determination and interaction of such broad economic aggregates as national income and product, consumers' expenditures and savings, producers' output of products and producers' investment in capital, the level and composition (by age, sex, and region) of employment, and exports and imports by way of government revenues (taxes) and expenditures?

Section VIII: Monetary Policy.

IS-LM Money Market: Money supply: $M_S = \underline{M}$. The demand for money: $M_D = M_0 + M_1 Y - M_2 r$.

Keynesian theory recognizes constraints such as liquidity traps and stresses active monetary policy in conjunction with fiscal policy to control demand and employment in the short term.

Believing that too much money is the main cause of inflation, monetarist theory promotes a rules-based strategy that emphasizes consistent money supply growth to maintain long-term price stability.

Given the course's focus on IS-LM and AD-AD models of an economy, Keynesian monetary theory was emphasized.

Insufficient aggregate demand (AD) can lead to long-term unemployment because sticky wages and prices impede markets from automatically clearing. Interest rates have an impact on consumers' demand for money (and savings), unless very low rates lead to a liquidity trap (cash hoarding). Depending on the MEC's elasticity, cutting interest rates and expanding the money supply can both encourage investment.

Section IX: The External Sector.

If a countries exports exceed imports, (X - M) > 0, consumption and investment spending is less than income. If exports are less than imports, (X - M) < 0, consumption and investment spending exceed income.

My online basic Keynesian IS(P)-LM(P) model with exports and imports is located at the URL: https://www.egwald.ca/macroeconomics/basicislmexternal.php.

In the open economy model on this web page, the rate of exchange and the level of prices are not determined endogenously. Their values can be set so that the economy's equilibrium varies with these exogenous variables. The parameters of the net export function can also be varied (within limits).

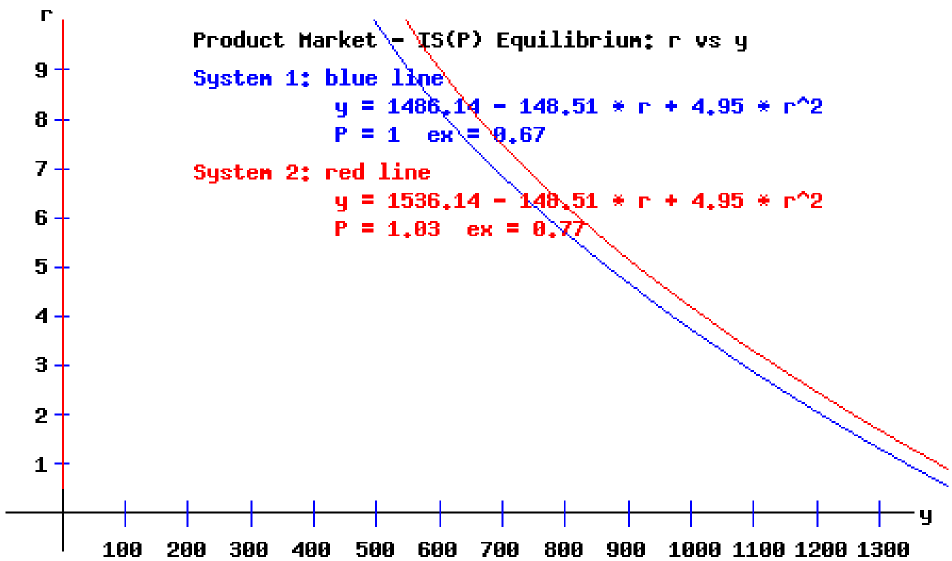
The estimates for the net export function are based, partly, on regression estimates for the Canadian Economy over the years 1982-2000. I was unable to estimate straightforward, distinct export and import functions because of the complicated ways that price level differences and exchange rate fluctuations across nations impact imports and exports.

The price level and exchange rate factors affect net exports; therefore, they also affect the overall demand for the commodity. As a result, variations in these factors cause a shift in the product market equilibrium. Similarly, because the demand for money is determined by the level of national income and the interest rate, and the real supply of money is determined by the price level, changes in the values of the price level and exchange rate variables also affect the money market equilibrium.

The Product Market – IS(P) Demand Equilibrium.

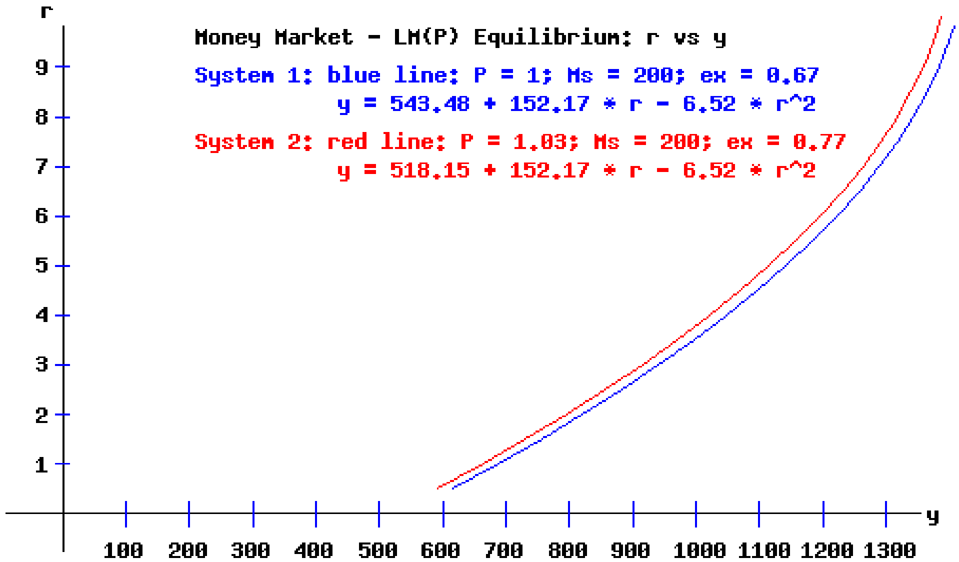
Consumers	$c = 100 + 0.7*yd - 20*r + 0.5*r^2$
Producers	$i = 100 + 0.2*y - 40*r + 1.5*r^2$

External Sector	System 1: netxp = 320 - 60 * P - 130 * ex - 0.15 * y System 2: netxp = 355 - 60 * P - 130 * ex - 0.15 * y
Government Expenditures	g = 210
Government Revenues	t = -25 + 0.22*y
Consumer Disposable Income	yd = y - (-25 + 0.22*y)
IS(P) Curve	c + i + g + netxp = y



The Money Market – LM(P) Equilibrium.

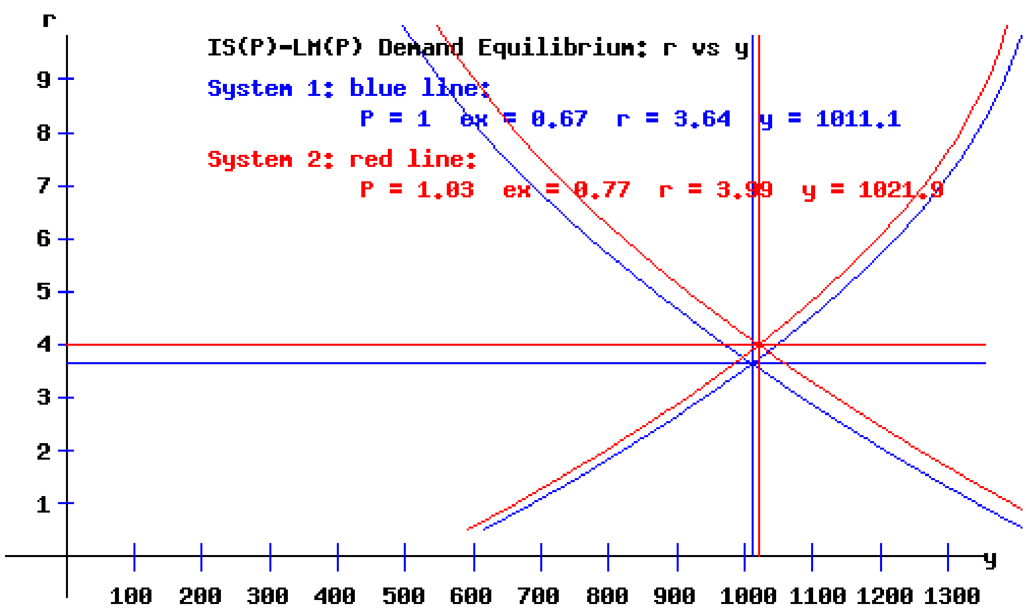
Consumers & Producers	md = 75 + 0.23*y - 35*r + 1.5*r^2
Nominal Money Supply; Real Money Supply	Ms = 200 ; ms = Ms / P
LM(P) Curve	md = ms



For this basic model of the Canadian economy, circa 2003, the equilibrium obtains where:

System 1: y = 1011.06, r = 3.64 , P = 1 , ex = 0.67.

System 2: y = 1021.92, r = 3.99 , P = 1.03 , ex = 0.77.



Economy Equilibrium Macroeconomic Variables and Aggregates.

	Blue System 1	Red System 2
National Income: y	1011.1	1021.9
Rate of Interest: r	3.64	3.99
Price Level: P	1	1.03

Exchange Rate: ex	0.67	0.77
Net Exports: netxp	21.24	39.81
Disposable National Income: yd	813.62	822.1
Consumer Expenditures: c	603.35	603.56
Firm Investments: i	176.46	168.55
Government Expenditures: g	210	210
Government Revenue: t	197.43	199.82
Nominal Money Supply: Ms	200	200
Real Money Supply: ms	200	194.17
Consumer Savings: s = yd - c	210.27	218.54

Perfect Capital Mobility.

Required reading was the important article by Robert Mundell, "Capital Mobility and Stabilization Policy under Fixed and Flexible Exchange Rates," Canadian Journal of Economics and Political Science November 1963.

His paper shows how exchange rate regimes and capital mobility essentially influence how effective fiscal and monetary policies are. When exchange rates are fixed, monetary policy preserves external balance while fiscal policy stimulates output; when exchange rates are flexible, monetary policy is strong but fiscal policy is neutralized. These insights, formalized in the Mundell-Fleming model, reshaped economic thought and remain critical for analyzing open economies.

The balance of payments (BoP) curve is a crucial idea in the Mundell-Fleming model. It is a horizontal line, interest rate $r = \underline{r}$, imposed on the interest rate vs income IS-LM model. It represents the situation in which there is no net inflow or outflow of foreign reserves required to balance international transactions, since the total of the current account (net exports) and the capital account (net capital flows) equals zero.

With fixed exchange rates, perfect capital mobility ensures the domestic interest rate equals the world interest rate for balance of payments equilibrium.

Monetary policy cannot shift the LM curve permanently, because an increase in the money supply shifts the LM curve to the right lowering the interest rate, $r < \underline{r}$. Capital outflows and a balance of payment deficit will put downward pressure on the exchange rate and cause the Central Bank to reduce the money supply, shifting the LM curve to the left to re-achieve the $r = \underline{r}$ equilibrium

An increase in government expenditure will shift the IS curve to the right causing the interest rate to rise, $r > \underline{r}$. Capital inflows will result in a balance of payments surplus putting upward pressure on the exchange rate. The central bank buys foreign reserves, increasing the money supply and shifting the LM curve to the right, until the $r = \underline{r}$ equilibrium is re-achieved and income, Y, has increased.

With flexible exchange rates, the exchange rate adjusts to ensure BoP equilibrium.

An increase in the money supply shifts the LM curve to the right, with $r < \underline{r}$, causing capital outflows and decreasing the exchange rate. An increase in net exports will shift the IS curve to the right. In equilibrium, $r = \underline{r}$, and income, Y, has increased.

An increase in government expenditure will shift the IS curve to the right, causing the interest rate to rise, $r > \underline{r}$. Capital will flow inward and the exchange rate will increase, reducing exports and shifting the IS curve to the left until the $r = \underline{r}$ equilibrium is re-achieved.

Professor Evans presented several qualitative comparative statistics models that illustrate the Mundell-Fleming's findings.

Section X: Prices, Wages, Unemployment, Inflation, etc.

The economic concept known as the Phillips Curve claims that unemployment and inflation have an inverse relation in an economy. Advocated by A.W. Phillips in 1958, it was supported by empirical evidence from the UK that showed a positive correlation between wage inflation and decreased unemployment rates. When unemployment is low, workers have greater negotiating power, which raises wages. Businesses pass the higher wages on to customers, causing inflation. Alternatively, excessive unemployment lowers inflation by reducing wage pressure.

In the short term, the government can either lower inflation at the expense of higher unemployment or lower unemployment at the expense of higher inflation by implementing expansionary fiscal or monetary policy.

In the long run, there is no trade-off because inflation expectations eventually catch up to real inflation, according to Milton Friedman and Edmund Phelps. Efforts to lower unemployment below its natural rate will cause inflation to rise faster.

I examined the role of Canadian unions in wage inflation in 1991. The Canadian economy was sneaking out of a recession. A prolonged recovery was doubtful unless corporate profits and investment by business and industries also recovered. This was unlikely if unions forced large wage settlements on employers, chocking off profitable investments.

Approximately 34 per cent of Canada’s labour force was unionized. That, plus the integrated nature of Canada’s industries provided unions with powerful leverage in contract negotiations, using strikes and the threat of strikes. Using regression analysis, I estimated that an extra one million of person days lost in a given year through labour disputes usually translates into a one per cent additional increase in wage agreements. Trade union militancy pays off.

Section XI: Getting It all Together: Can one Reconcile Keynes and the Neo-Classicists?

Axel Leijonhufvud in *On Keynesian Economics and the Economics of Keynes* posits that while in general equilibrium models supply and demand are functions of prices, in Keynesian models supply and demand are functions of income and interest rates.

As part of my enrollment in Economics 502, I wrote a paper, "Review of Disequilibrium in Markets: The Micro-Foundations," URL: https://www.egwald.ca/macroeconomics/pdf/foundationdisequilibrium.pdf. It presents a model that lays the micro-foundation for Keynesian models using the concept of disequilibrium in markets.

The model assumes that goods have been allocated to economic agents, and / or can be produced by agents. It examines the issue of how the goods will be re-distributed through exchange with other agents when market prices do not equate the demand and supply of goods instantaneously, as in the general equilibrium models. Agents could be constrained selling or buying goods and / or choose to limit production, when markets do not clear demand with supply.

An agent solves a Static Recursive Algorithm while undertaking transactions, a procedure that is a generalization of Robert Clower's Dual Decision Hypothesis set out in "The Keynesian Counter-Revolution: A Theoretical Appraisal." For further information, see my paper.

Clower uses a disequilibrium choice theoretic framework as a basis in deriving Keynes' consumption function. In the classical analysis, the representative household regards utility maximization as subject only to the budget constraint. This implies that the notional consumption function does not depend on income, because income (amount of labor it will supply) and consumption are chosen simultaneously.

If labor is in excess supply, the household must include the labor constraint when it maximizes utility. Thus, the effective consumption function will have effective income (i.e. the amount of labor it can sell) as one of its arguments. But this implies that effective demand in the commodity market will be less than the notional demand. That is, the excess supply in the labor market has contributed to excess supply in the commodity market.

Grossman and Barro, "A General Disequilibrium Model of Income and Employment," analyze further the effect of disequilibrium arising out of excess demand. Using the disequilibrium choice theoretic framework, it follows that if there is excess demand for labor, the firm will either increase its demand for capital and/or produce less. If there is excess demand for commodities, the household will either increase its money balances and/or decrease its supply of labour.

The conclusions obtained by these authors are consistent with my Static Recursive Algorithm.

According to neo-classical theory of demand for investment, a profit maximizing firm desires an optimal stock of capital, based on its state of expectations about the prevailing economic conditions. If its capital stock is less than optimal, it invests. Conversely, if its capital stock is greater than optimal, it disinvests.

The income accelerator theory of investment demand, on the other hand, regards output as exogenously determined, with investment an increasing function of changes in the level of output.

In "A Choice-Theoretic Model of an Income-Investment Accelerator," Grossman provides "a reconciliation of the neoclassical and accelerator theories of investment demand." His approach is similar to the disequilibrium models above, although he is operating in an intertemporal context. If the market for output does not clear at the existing price, output to a profit maximizing firm is no longer a decision variable. Thus, effective "investment demand will become a function of the level of output which it will be able to sell."

Grossman shows that the assumption of static expectations about the output constraint implies a gradual (flexible) income-investment accelerator. On the other hand, the assumption of static expectations about the rate of change of constrained output implies an instantaneous accelerator relationship. Grossman concludes that profit maximization implies that if the output market is in disequilibrium, the income-accelerator approach is the correct theory of investment demand; if the output market is in equilibrium given the state of expectations, the neoclassical approach is the correct theory of investment demand.

I shall revisit my disequilibrium model in the Economics 503 Macroeconomic Dynamics section, Equilibrium theories of Keynesian unemployment.

Economics 502: Conclusions.

The IS-LM model and its extensions are an important place to start when learning about how economies function. The instructor, Dr. Evans, worked with John Helliwell, his colleague at UBC, on the RDX macroeconomic models for the Bank of Canada. Helliwell, who worked as a consultant for the Bank of Canada from 1965 to 1980, focused on real-world econometric modeling applications in his work on the RDX1, and RDX2 models. The Keynesian IS-LM framework and the Phillips curve served as the theoretical foundation for these models.

These macroeconomic modeling conventions were represented in the material covered in Dr. Evans' course. The course sections on consumption, capital and investment, fiscal and monetary policy, and the external sector provided a thorough explanation of the theoretical underpinnings of the IS-LM model's equations.

Chapter 15 of W. H. Branson's macroeconomics textbook, The Foreign Sector and the Balance of Payments, would have given me a better grasp of the foreign sector of the economy at that time. In 1970, Canada adopted a system of variable exchange rates. The United States and other nations did not allow global market forces to determine their currency rates until the Bretton Woods arrangement fell apart in 1973.

AI Groc's overview of the RDX2 Model and Its Theoretical Context.

"The RDX2 model, developed by the Bank of Canada in the late 1960s and published in 1971, was a large-scale quarterly econometric model with 142 stochastic equations, designed to capture the dynamics of the Canadian economy. It built on earlier models (RDX and RDX1) and incorporated a mix of Keynesian and neoclassical elements. The theoretical basis included:

- **IS-LM Framework**: RDX2 modeled the interaction between the goods market (investment and saving) and the money market (liquidity preference and money supply), similar to the IS-LM model, but with greater disaggregation and dynamic adjustments.
- **Phillips Curve**: The model included wage and price dynamics, reflecting trade-offs between inflation and unemployment, consistent with Phillips Curve relationships.

- **Quantity Theory Influences**: Money supply and monetary policy played significant roles, with equations linking monetary variables to expenditure and prices.
- **Open Economy Features**: Unlike the closed-economy IS-LM model, RDX2 accounted for international trade, capital flows, and exchange rates, aligning with the Mundell-Fleming extension of IS-LM.

The model was used for policy simulations, forecasting, and analyzing monetary and fiscal policy impacts, particularly under Canada's flexible exchange rate regime post-1970.

Example Equations in RDX2.

"RDX2's equations are highly detailed and disaggregated compared to the simplified functions in textbook IS-LM models. Below, I provide examples of the consumption and investment functions, adapted from *The Structure of RDX2* (Helliwell et al., 1971) and contextualized with IS-LM parallels.

Consumption Function.

In RDX2, the consumption function is more disaggregated, distinguishing between durable, semi-durable, non-durable goods, and services. It incorporates dynamic elements (e.g., lagged variables) and additional determinants like wealth and interest rates, reflecting a more realistic Canadian economy. A representative consumption function for non-durable goods in RDX2 might take the following form (stylized for clarity):

$$CND = a0 + a1\,Yd + a2\,W + a3\,CND_{-1} - a4\,r$$

where:

CND: Real consumption expenditure on non-durable goods,
a0: Constant (autonomous consumption component),
Yd: Real disposable income (Y−T, adjusted for inflation),
W: Real household wealth (capturing wealth effects),
CND_{-1}: Lagged consumption (capturing habit persistence or adjustment lags),
r: Real interest rate (reflecting the cost of borrowing or opportunity cost of spending),
a1 ,a2 ,a3, a4: Parameters estimated econometrically.

Investment Function.

In RDX2, investment is disaggregated into business fixed investment, residential construction, and inventory investment, with equations incorporating accelerator effects, interest rates, and output gaps. A representative business fixed investment function in RDX2 might look like (stylized):

$$IBF = b0 + b1\,\Delta Y + b2\,(Y - Y^*) - b3\,r + b4\,K_{-1}$$

where:

IBF: Real business fixed investment,
b0: Constant (autonomous investment component),
ΔY: Change in output (accelerator effect, investment responds to growth in demand),
$Y - Y^*$: Output gap (actual output Y minus potential output Y^*, capacity utilization),
r: Real interest rate (cost of capital),
K_{-1}: Lagged capital stock (reflecting existing capital influencing new investment),
b1, b2, b3, b4: Estimated parameters."

Thank you, Grok.

Government Economic Incentives.

In 1990, my letter to the editors of a number of newspapers was published.

“The annual payroll of the federal government has more than doubled over the previous decade. What have our politicians, bureaucrats and central bankers provided for this largess?

Canada’s gross national product has increased by 38 percent over the last decade, but GNP per employed person has only increased by 12 per cent, the remaining 26 per sent due to the 23 per cent increase in the number of people employed.

Consumer prices have increased by 70 per cent with another jolt expected soon from the GST and rising world oil prices.

Right now, GNP is falling, the rates of inflation and unemployment are rising and the country is awash with uncertainty about economic prospects because of the GST, high rates of interest, a huge federal government debt and disunity from the Meech Lake Accord.

Do we need economic incentives to ensure that Canada’s policy makers provide us with a steady rate of growth of GNP and a low rate of inflation?

Suppose the total payroll of the federal government was adjusted annually by the change in GNP less the rate of inflation (deflation) for that year. Thus, if the GNP increased by five per cent and the rate of inflation was three per cent, then the total annual payroll of the federal government would increase by two per cent.

You can be sure that very quickly policies would be formulated and implemented to maximize GNP growth and employment with little or no inflation.

Let’s use economic incentives on our politicians, bureaucrats and central bankers to give us what we want.”

Economics 503: Economic Fluctuations and Growth.

Dr. Keizo Nagatani, the UBC instructor for in the spring of 1973, taught this course for about a decade. His lectures were updated as new theories and empirical studies of the economic dynamics of modern economies were published. My comments are based on his lectures as published in 1981 by Cambridge University Press with the title, *Macroeconomic Dynamics.*

He recommended that I read up on the Arrow-Debreu model of a competitive economy (https://www.jstor.org/stable/1907353?origin=crossref) when I paid him a visit at his UBC office prior to the commencement of the course. I had already completed this while enrolled in the Theory of Games and Programming graduate mathematics course taught by Dr. Rodrigo Restrepo in 1967–1968.

I also studied the book, *Theory of Value*, written by Gerrard Debreu in 1959, in an effort to find new insights. Since I had completed Dr. Maurice Sion's undergraduate and graduate mathematics courses in real analysis and measure theory for my M.Sc. degree in 1969, I felt comfortable with the mathematical level of Debreu's work.

I used this mathematical background in my Ph.D. thesis, “Money as a Transaction Technology: A Game Theoretic Approach” (https://www.egwald.ca/wiens/elmerwiensphdthesis.pdf) with Dr. Nagatani as thesis supervisor.

“A barter and a monetary economy are modelled using the cooperative game approach. The feature that distinguishes the two economies is the manner in which exchange activities are organized in the face of transaction costs. While division of labour or specialization is exploited in the monetary economy’s technology of exchange, it is not exploited in that of the barter economy. The presence of a medium of exchange in the monetary economy permits its specialized traders to operate efficiently.

The cooperative game approach admits group rationality along with the usual assumption of individual rationality. Group rationality means that individuals are able to perceive their independence. Money is explained as the product of the interactions of between individual rationality (utility maximizing consumers and profit maximizing traders) and group rationality (the ability to perceive the benefits of monetary exchange versus barter exchange). Consequently, money is not viewed as an object, but as an institution. Its value reflects the relative superiority of a monetary economy over a barter economy. “

In 2005, I revisited the issue of the origin of money, and the difference between monetary and barter economies when I took Dr. Lorraine Weir’s English Course, First Nations Studies. My paper, “Linguistic and Commodity Exchanges” (https://www.egwald.ca/ubcstudent/aboriginal/exchanges.php), concludes with the paragraph:

“There is a structural difference between barter and monetary commodity exchanges, and between oral and written linguistic exchanges. While barter and oral exchanges may take place simultaneously with monetary and written exchanges in a community, it is money and writing that permit the structural differences in the technology of exchange. With money and writing / reading, the superior technologies of commodity and linguistic exchange based on division of labour become feasible. Moreover, the advantages of monetary over barter exchanges provide a powerful motivation for the use of writing in the linguistic exchanges within the technology of commodity exchange.”

Economics 503: Syllabus, Sections, Course Notes.

The five pages of Dr. Nagatani's course syllabus were broken up into twelve sections, with a total of about sixty-five recommended readings. Each section’s content was covered in a class handout. With edits for clarity and expositional rigor, their substance essentially follows the parts of Nagatani's book, *Macroeconomic Dynamics*. The instructor supplemented the handouts with in-class verbal and blackboard explanations.

Section 1. A Historical Review.

After reviewing the contributions of the Mercantilist, Classical, Neoclassical, and Keynesian economists, Nagatani rejected the UK Cambridge economists, including Geoff Harcourt and Joan Robinson, with their class fixation on capitalists, labourers and the accompanying wages fund.

Nagatani stated the objective of the course as follows:

"While Neoclassical theory was essentially a static theory and remained poor in capital theory for a long time, we now know alternative ways of producing interesting dynamic models within the Neoclassical setting. This subject will be pursued in the rest of the course."

His textbook states: "How to keep entrepreneurs optimistic and willing investors is the *sin qua non* of macroeconomic policy." Furthermore, macroeconomic theories had to take into account Keynes' emphasis on the central role asset markets, the activities of entrepreneurs and saver-investors, in a macroeconomy.

With the internet and easy access to stock markets, spot and future commodity markets, bond markets, etc., we can all be entrepreneurs now. Futures contracts are traded on listed exchanges for equities, oil and gas, gold and silver, currencies, and agriculture products. Forward commodity contracts, negotiated privately between the buyer and seller, are an agreement to deliver a quantity of a specific asset at a future date at a given price.

Section 2: Dynamic Optimization.

Nagatani juxtaposes the classical constrained static-optimization programming problem with the dynamic-optimization problem. In the former, the solution vector and co-state vector's components are real numbers; in the latter, real-valued functions of time. The Weierstrass theorem and its generalization to function spaces provide the existence of solution vectors. Nagatani solves a simple, qualitative model of optimal economic growth, and illustrates its solution graphically.

An understanding of undergraduate mathematics courses in ordinary, partial, and nonlinear differential equations as well as fourth-year courses in the calculus of variations and optimal control theory is recommended in order to model dynamic economic processes.

One Dimensional Dynamics: Continuous Time Models.

The dynamics and bifurcations (and terminology) of one-dimensional continuous time models are discussed in detail on the webpage at the URL:
https://www.egwald.ca/nonlineardynamics/onedimensionaldynamics.php.

One-dimensional differential equation.

The general form of a one-dimensional differential equation with a parameter r is:

(1) $$dx / dt = f(x, r), x(0) = x0.$$

If the function f is linear in x, the differential equation (9) is called linear; if the function f is nonlinear in x, it is called nonlinear. Determining the explicit solution of most nonlinear differential equations is very difficult or impossible. Consequently, the geometric properties of the function f are investigated to obtain qualitative information about the solution to the differential equation.

Fixed Points and Stability.

For a specific value of the parameter r = r1, write:

(2) $$dx / dt = f(x, r1) = f(x).$$

A fixed point of f is a number x^* such that: $f(x^*) = 0$. A fixed-point x^* can be an equilibrium point, a critical point, a stationary point, or a rest point.

A fixed-point x^* is stable if all trajectories starting "sufficiently close" to x^* stay "close" to x^*. The fixed-point x^* is asymptotically stable if it is stable and trajectories starting "sufficiently close" to x^* eventually approach (decay) to x^* as $t \rightarrow$ infinity. A fixed point that is not stable is called unstable.

Linear Stability Analysis.

One can analyze the local stability of the differential equation (2) about a fixed point by examining the first derivative of the function f with respect to x evaluated at x^*: $\lambda = f'(x^*)$. The slope of the function f at the fixed-point x^* determines the stability of (2). The nonlinear process determined by the function f is stable at x^* if: $\lambda = f'(x^*) < 0$; the nonlinear process is unstable at x^* if: $\lambda = f'(x^*) > 0$. If $\lambda \neq 0$, x^* is called a **hyperbolic fixed point**. If $\lambda = 0$, x^* is called a **non-hyperbolic fixed point**. The stability of the process must be examined for each situation if: $\lambda = f'(x^*) = 0$.

One Dimensional Bifurcations.

A nonlinear process experiences a qualitative change in its dynamics when its fixed-points change in nature. When a change in a parameter result in a qualitative change in the dynamics of a nonlinear process, the process is said to have gone through a bifurcation.

Bifurcations are classified by the way in which the fixed points of the function f change their number, location, form, and stability. It is convenient to consider normal forms for the different types of bifurcations, and the conditions under which the dynamic process described by equation (1) will experience a specific type of bifurcation.

Solow Growth Model.

The famous Solow Growth Model analyzed on the webpage at the URL:
https://www.egwald.ca/nonlineardynamics/onedimensionaldynamics.php#neoclassicaleconomicgrowth
has a **transcritical** bifurcation. The time paths of capital stock, income and consumption depend on the value of s, the rate of saving out of income. The analysis determines the critical value of s, $\mathbf{s_C}$. There is one stable equilibrium if $s = \mathbf{s_C}$; two equilibria (stable, unstable) if $s > \mathbf{s_C}$.

Labour Market Bifurcation. The Backward Bending Labour Supply Curve.

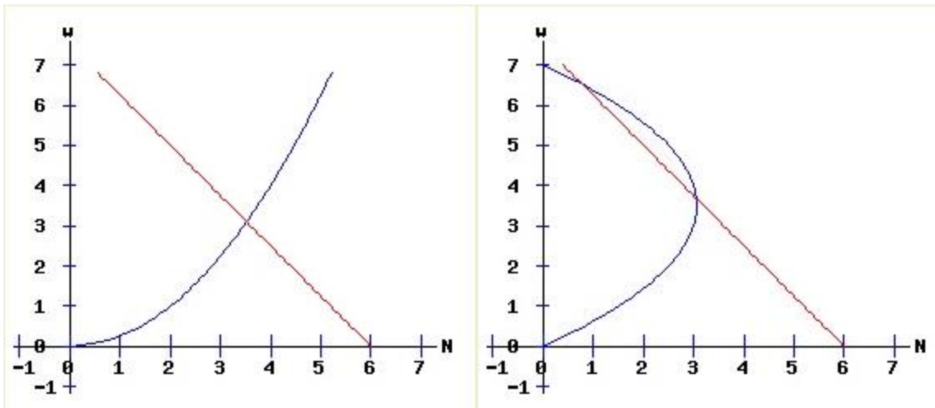
The labour market has a demand and a supply schedule. The demand for labour, L_d, by firms is a decreasing function of the wage rate, w, while the supply of labour, L_s, by workers is an increasing function of the wage rate, w. The first diagram below illustrates these relationships, where the intersection of the L_d and L_s curves determine the equilibrium wage rate, w, and equilibrium number of workers hired, N. In the diagram of backward bending labour supply, the supply of labour function L_s, increases with the wage rate to a wage $\underline{w}$, and decreases with any further increases in w.

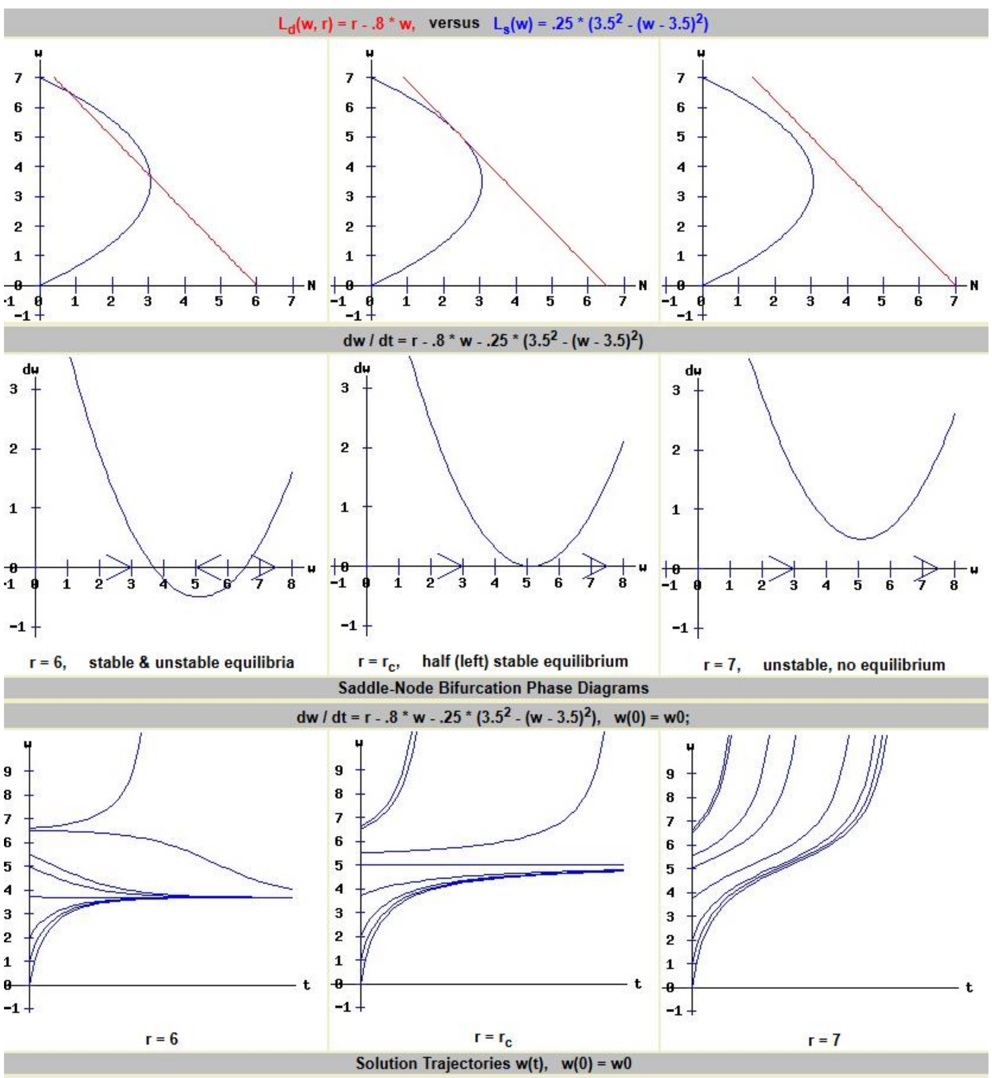
To illustrate the model geometrically, assume the functional forms:

$$L_d(w, r) = r - .8 * w, \quad L_s(w) = .25 * (3.5^2 - (w - 3.5)^2),$$

where r is a parameter, and $L_s(w)$ is the backward bending labour supply schedule. The wage rate, w, adjusts according to the differential equation:

dw / dt = f(w, r) = L_d(w, r) - L_s(w), w(0) = w0.





Step 1: Determine the non-hyperbolic fixed points. Solve the two linear equations for w = w^* and r = r_c:

SN1: f(w, r) = r - .8 * w - .25 * (3.52 - (w - 3.5)2) = 0, SN2: $fw(w, r) = -.8 -.25 * (-2 * (w - 3.5)) = 0 \rightarrow w^* = 5.1 \rightarrow rc = 6.5025.$

The non-hyperbolic fixed point is: $(w, r) = (w^*, r_c) = (5.1, 6.5025)$.

Step 2: Determine the non-zero a and b coefficients.

SN3: $f_r(w, r) = 1 \rightarrow a = f_r(x^*, r_c) = 1 > 0,$

SN4: $f_{w,w}(w, r) = .25 * 2 \rightarrow b = 0.5 * f_{w,w}(x^*, r_c) = 0.25 > 0$

Since $a > 0$ and $b > 0$, the dynamic process of the labour market has a Type One saddle-node bifurcation at $w^* = 5.1$, $r_c = 6.5025$.

Section 3: Hicksian Temporary equilibrium.

A limitation of qualitative comparative static optimization models is that they can only indicate the direction of change of an endogenous variable at equilibrium due to a variation in an exogenous parameter. Qualitative comparative dynamics models in optimal control theory have similar limitations. The qualitative time-dependent path or profile of an endogenous variables in a dynamic system is dependent upon the time-dependent path of the exogenous variables.

The time trajectories of endogenous variables in relation to changes in the time paths of exogenous variables are revealed by a quantitative model of the dynamic system. Examples of Optimal Control Theory applications using the Pontryagin Maximum Principle are online are on my webpage at the URL: https://www.egwald.ca/optimalcontrol/index.php.

Section 4: Rational-Expectations Models.

The "rational expectations" modeling technique posits that individuals form expectations based on the most accurate information available and adjust them to avoid systematic errors.

The Stockholm School of Economics of the 1920's and 30's anticipated certain theoretical concepts that prevail in modern economics. The subsequent three paragraphs are extracted from my graduate student work titled "Monetary Equilibrium and the Stockholm School," available at the URL: https://www.egwald.ca/macroeconomics/pdf/monetaryequilibrium.pdf.

"The first concept, due to Wicksell, was the "aggregate demand-supply approach—emphasizing the relation of investment to savings—to changes in value and associated changes in tempo and scope of economic activity which find expression in fluctuations of price levels, income, and employment" (Uhr 1951). Wicksell rejected Says law—that supply creates it own demand—and the resulting bifurcation of theoretical economic analysis into the theory of relative prices and the theory of money.

The second concept, presented by Myrdal in 1927, was the incorporation of expectations into economic theory. He is credited with distinguishing between *ex ante* (looking forward) and *ex post* (looking backward) variables in dynamic economic analysis.

The third concept, first alluded to by Myrdal (1927), but developed by Lindahl (1929), and extended by Lundberg (1936) and Hicks (1939) was the "method of analyzing a dynamic process as a series of equilibria, between which there occur unforeseen events with consequent gains or losses," Lindahl (1939)."

Individuals, firms, and governments form expectations, looking forward, based on information available, looking backward. These economic agents act in this way to optimize their well-being, profits, and prospects to remain in power by way of re-election or otherwise.

Nagatani states that "the central goal of macroeconomics is to understand economic fluctuations. If all agents were competent general-equilibrium theorists, data collectors, and econometricians . . . economic fluctuations would be eliminated."

Kaldor Business Cycle Model.

As shown by Nicholas Kaldor's business cycle model, by balancing supply and demand schedules, markets themselves can set off economic cycles, if the market adjusts too quickly, https://www.egwald.ca/nonlineardynamics/limitcycles.php#kaldormodel.

Nicholas Kaldor investigated the interaction between investment and savings to produce cyclical movements in income and capital. In his model, investment and savings are functions of the flow of income, **y**, and the stock of capital, **k**, with capital depreciating at a constant rate, **δ**. The flow of income equals the value of the production of consumer and capital goods. The dynamics of the system are governed by:

$$dy/dt = f(y, k) = \alpha * (I(y, k) - S(y, k)), \quad (1)$$

$$dk/dt = g(y, k) = I(y, k) - \delta * k. \quad (2)$$

The parameter **α** in equation (1) determines the speed at which the aggregated market for capital and consumer goods adjusts. Equation (1) describes the dynamics of income, which increases when investment, I(y, k), exceeds savings, S(y, k), and vice versa. Equation (2) describes the growth of the stock of capital, equalling the production of capital goods, I(y, k), less depreciation, $\delta * k$.

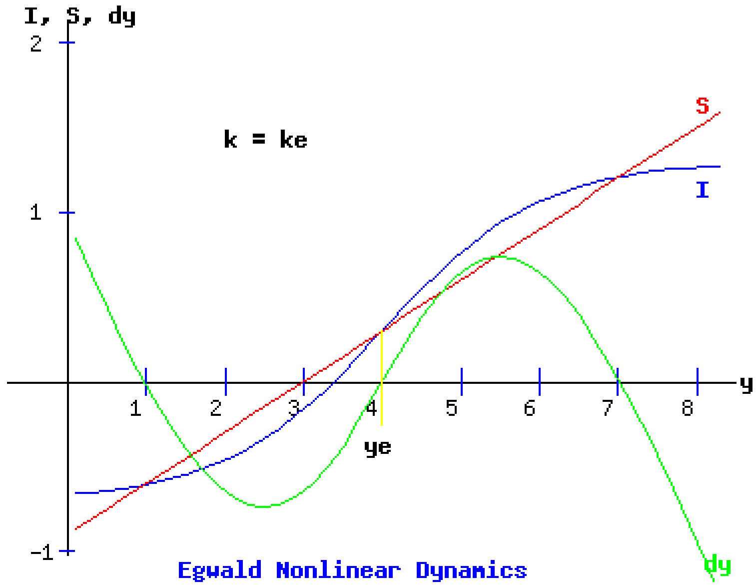
Suppose (y_e, k_e) is the fixed point of the Kaldor system, so that:

$$f(y_e, k_e) = \alpha * (I(y_e, k_e) - S(y_e, k_e)) = 0,$$

$$g(y_e, k_e) = I(y_e, k_e) - \delta * k_e = 0.$$

The diagram below graphs **investment**, **savings**, and their difference, **dy**, as functions of income, y, for capital at $k = k_e$, with $\alpha = 4$. As shown in the diagram, investment and savings are increasing functions of income, y. Furthermore, the slope of the investment function, I_y, increases for $y < y_e$, while I_y decreases for $y > y_e$. The amplitude of dy depends on α.

For low values of y, savings are negative as people liquidate wealth. Investment is also negative for low values of y. Capital (machinery and equipment) taken out of the production processes may be too obsolete, costly, or unprofitable to put back on-line when demand for product increases. Furthermore, new product lines may require new machinery and equipment as demand for consumer goods increases as income increases. (Kaldor assumed that investment is positive for all y > 0.)



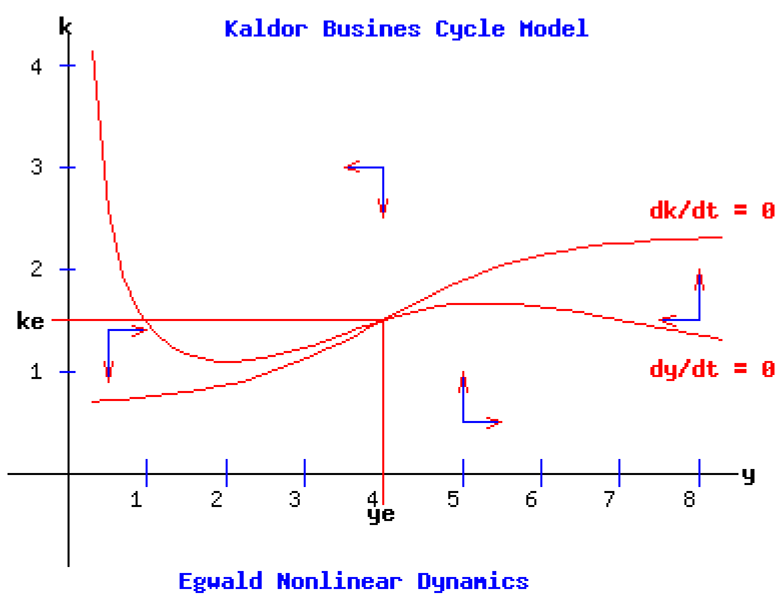
In the spirit of Kaldor, the functions I(y, k) and S(y, k) are restricted as follows:

$I_y > 0$; $S_y > 0$; $I_k < 0$; $S_k > 0$

$I_{yy} > 0$ for $y < y_e$, and $I_{yy} < 0$ for $y > y_e$

$I_y(y_e, k_e) > S_y(y_e, k_e)$

With these assumptions, the phase portrait and the y and k nullclines appear as follows:



The stability of the fixed point depends on the Jacobian of the system:

$$J(y, k) = \begin{pmatrix} \alpha * (I_y - S_y) & \alpha * (I_k - S_k) \\ I_y & I_k - \delta \end{pmatrix}$$

evaluated at the fixed point (y_e, k_e).

For the specific functions used here, $(y_e, k_e) = (4, 1.5)$, and the Jacobian is:

$$J(y, k) = \begin{pmatrix} \alpha * (1/2 - 3/10) & \alpha * (-1 - 1/5) \\ 1/2 & -1 - 1/5 \end{pmatrix}$$

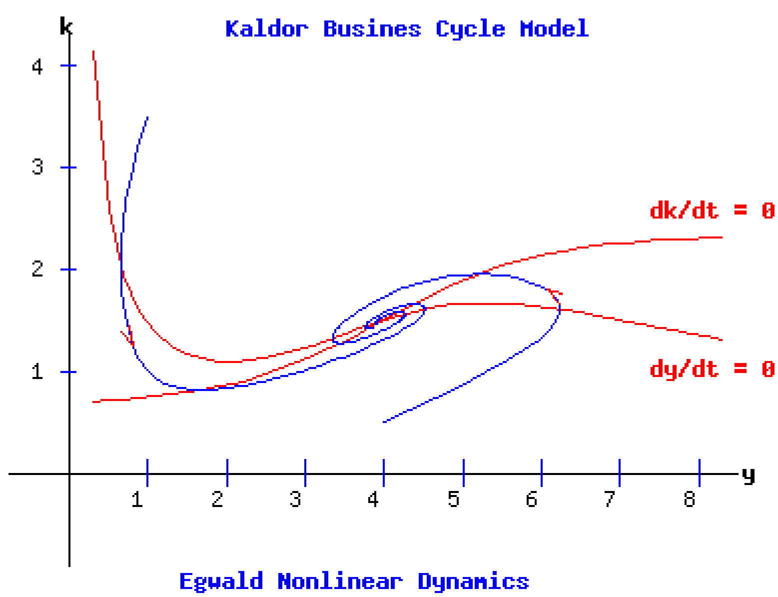
Consequently, the systems stability is determined by:

$$p = \text{trace}(J) = \alpha * (I_y - S_y) + (I_k - \delta) = 1/5 * \alpha - 6/5,$$

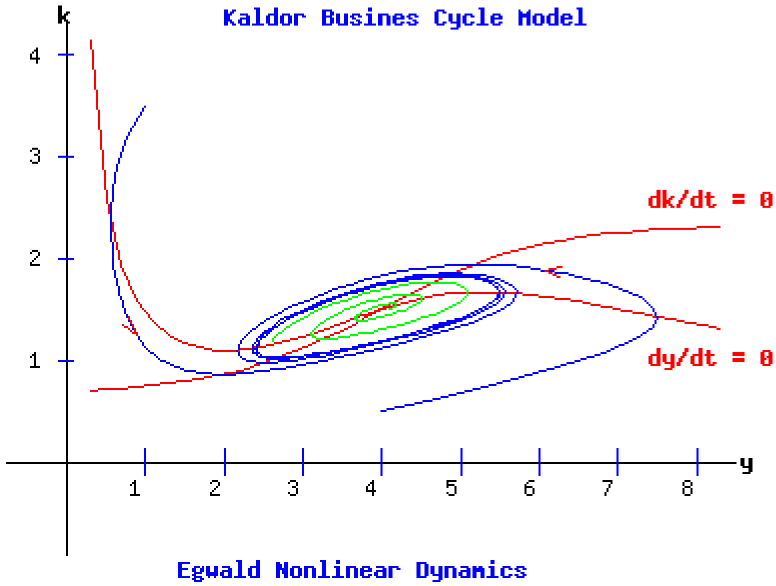
$$q = \det(J) = \alpha * (I_y - S_y) * (I_k - \delta) - I_y * \alpha * (I_k - S_k) = 4/25 * \alpha > 0$$

The sign of p is negative for $\alpha < 6$, and positive for $\alpha > 6$.

For $\alpha < 6$, the fixed point is an attractor, as shown in the next diagram:

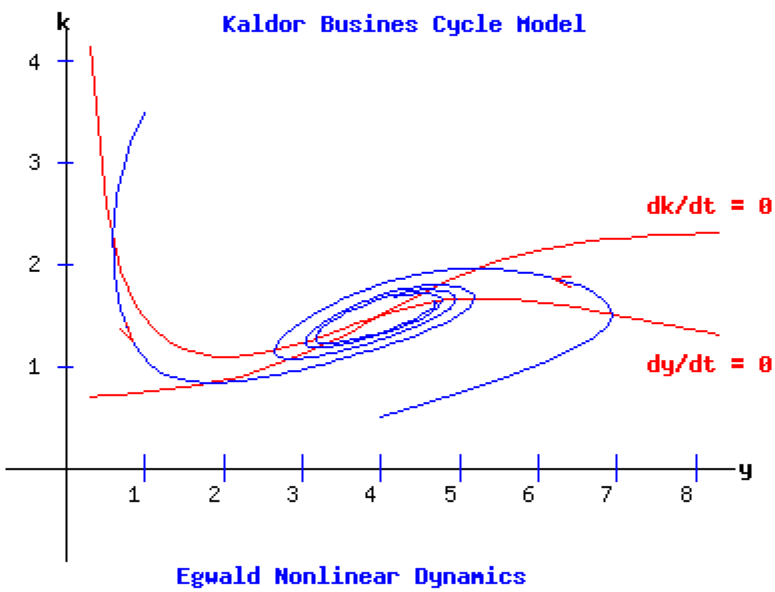


For $\alpha > 6$, the fixed point is a repeller. Moreover, as shown in the next diagram a limit cycle emerges:



Kaldor Model Hopf Bifurcation

The fixed point of the Kaldor model loses its stability as α, the adjustment coefficient of the income-goods market equation, increases past $\alpha = \alpha_c = 6$. The fixed-point switches from attractor to repeller, and a limit cycle emerges for $\alpha > 6$. In the next diagram $\alpha = \alpha_c = 6$.



Chapter 5: Equilibrium Theories of Keynesian Unemployment.

Because this chapter focuses on Keynes and the Keynesians, I will go over each section.

Section 5.1 The Problem.

Professor Nagatani's published lectures assert that macroeconomic theories after Keynes have inadequately captured J. M. Keynes' views as articulated in *A Treatise on Money (1930)* and *The General Theory of Employment, Interest, and Money (1936)*.

Keynes, in his analysis of the 1930s Depression, acknowledged the potential for inadequate market demand. Nagatani writes that Keynes "analysis of demand led to his theory of effective demand and the notion of the aggregate-supply function. . . . In this connection, the most important contribution of Keynes was his emphasize on the role of expectations."

Investment demand is influenced by long-term expectations, Keynes marginal efficiency of capital, and the expense of financing investments at the prevailing interest rates in asset markets, including the stock market. Similar factors influence consumption demand.

Nagatani states that Keynes was of the opinion that "an interest rate was the price of renting the liquidity services of money and hence was directly influenced by the state of liquidity preference."

Aggregate demand, investment plus consumption, "determines the level of employment via the aggregate-supply function $Z = Ø(N)$." Z is the value of output. Quoting Keynes, "Z is the proceeds the expectation of which will induce a level of employment N." "These types of expectations were called 'short-term' expectations as against the 'long-term' expectations governing investment decisions.

In what ways have macroeconomic theorists inadequately understood Keynes' model "that embraces all the parts of the economy as an inseparable whole"? Nagatani goes on to state that the "central policy question is, of course, how to control the level of expected proceeds or the level of effective demand."

How to Incorporate an Aggregate Supply Schedule into the IS-LM Model.

My Classical and Keynesian Aggregate Demand – Aggregate Supply model develops the labour market with an aggregate production function to produce an aggregate supply function (URL: https://www.egwald.ca/macroeconomics/keynesian.php).

The Keynesian IS-LM model focuses on the "demand side" of the economy - the relationship between national income and the aggregate demand for product (goods and services) by consumers, producers, and governments. The IS-LM model ignores the price level of goods and services, the level of employment, the wage rate of workers, and the amount of product output. To include these macroeconomics aggregates in the model, I combined the labour market with the aggregate relationship between employment of workers and the level of product output, providing the aggregate supply equation - a relationship between the price level and the level of output produced by firms. I also modified the equations that describe the money market of the basic IS-LM model to reflect the effect that changes in the price level have on the perceived "real supply" of money, yielding the aggregate demand equation - a relationship between the price level and the demand for output of firms.

The intersection of the aggregate demand and aggregate supply equations will yield the equilibrium level of output, the price level, the wage rate, and the level of employment, along with the rate of interest and the values of all the other macroeconomics variables obtained from the IS-LM model. This aggregate demand-aggregate supply (AD-AS) economics model approximates the relationships among these key macroeconomics aggregates.

The aggregate production function relates the amount of output produced in the economy to the amounts of inputs used, the amounts of labour, capital, and materials & supplies actively employed. Labour, capital, and materials & supplies are called the factors of production. Increasing the amount of any factor of production, while holding constant the other factors, will increase the amount of output. In this basic short-run AS-AD model, I consider the effect on output of varying the amount of labour employed, assuming materials & supplies vary in direct proportion to labour. Capital (more properly, capital services) are constant over the relevant time period of the model. The aggregate production function specifies the relation between labour inputs and the output of the firms in the economy. Since the contribution to output of one hour of labour employed depends on the type of job, and on the experience and education of the person employed, I assume I can convert the sum of all labour inputs into one generic labour input.

System of equations for the AD-AS model.

The values of the parameters of the macroeconomics functions reflect the annual average values of the macroeconomics aggregates of the Canadian economy during the years of 1998 - 2003

Aggregate Supply Equations

Aggregate Production	ys = 51.37 + 45 * N - 0.25 * N^2 - 0.004117 * N^3
Labour Demand	Wd = P * (45 - 0.5 * N - 0.01235 * N^2)
Classical Labour supply	Ws = P * (0 + 0.5 * N + 0.01575 * N^2)
Keynesian Labour supply	Ws = 0 + 0.5 * N + 0.01575 * N^2
Labour Market Equilibrium	**Wd = Ws**

Aggregate Demand Equations

Product Market - IS Demand Equilibrium

Consumers	$c = 100 + 0.7*yd - 20*r + 0.5*r^2$
Producers	$i = 100 + 0.2*y - 40*r + 1.5*r^2$
Government Expenditures	g = 210
Government Revenues	t = -25 + 0.22*y
Consumer Disposable Income	yd = y - (-25 + 0.22*y)
IS Curve	**c + i + g = y**

Money Market - LM(P) Demand Equilibrium

Consumers & Producers	$md = 75 + 0.23*y - 35*r + 1.5*r^2$
Nominal Money Supply; Real Money Supply	$Ms = 200$; $ms = Ms / P = 200$
LM(P) Curve	**md = ms**

Consider the entrepreneurs or firm owners contemplating expanding their production in the next and subsequent years by acquiring more capital assets and employing more workers.

The entrepreneurs, cognizant of the present state of the economy using my AD-AS model, project their expectations regarding the profitability of investment given the expected prosperity, liquidity of money and asset markets, and growth of the economy. The savvy entrepreneurs consider how the economy might change, that is, the how the parameters of the AD-AS model might change over the lifetime of contemplated capital investment. They attempt to anticipate the activities of the government--fiscal and monetary policies, consumers, importers and exporters, and fellow entrepreneurs.

Comparative Statics of the Classical and Keynesian AS-As model.

Shifting the Economy's Equilibrium: https://www.egwald.ca/macroeconomics/compstatskeynesian.php

On this dynamic webpage the parameters of the Classical & Keynesian AD-AS models can be changed to see how the values of the equilibrium macroeconomics variables and aggregates change. By changing the parameters of the labour market and the aggregate production function, the economy's aggregate supply schedule shifts. Moreover, by changing the parameters of the product and money market, the economy's aggregate demand schedule shifts. The intersection of the aggregate supply and demand schedules determines the economy's equilibrium price level, and national income & product. Feeding these equilibrium values into the equations that underlie their schedules determines the level of employment, the money wage rate, the economy's wage bill, the interest rate, the demand for money, consumer expenditures, investment in capital by firms, and government revenues.

Section 5.2 The Dual-Decision Hypothesis.

What impact will insufficient effective demand have on economic agents' choices? What will be the impact on the markets for goods and labour skills if long-term expectations and overall demand are insufficient for full employment?

In Section XI of Macroeconomics 502, I outlined my paper, “Review of Disequilibrium in Markets: The Micro-Foundations,” URL: https://www.egwald.ca/macroeconomics/pdf/foundationdisequilibrium.pdf. My model lays the micro-foundation for some Keynesian models using the concept of disequilibrium in markets. For example, disequilibrium in the labour market can explain the Keynesian consumption function used in IS-LM models.

The Micro-Foundations of Disequilibrium

The model assumes that goods have been allocated to the agents, and / or can be produced by agents. It examines the issue of how the goods will be re-distributed through exchange with other agents when market prices do not equate the demand and supply of goods instantaneously, as in the general equilibrium models. Agents could be constrained selling or buying goods and / or choose to limit production, when markets do not clear demand with supply. With the model, we attempt to lay the micro-foundations of transactions arising out of market disequilibrium.

The Economy of "n" Markets and "m" Agents.

Assume the economy consists of:

1. n markets for goods labelled with the index i = 1 … n,
2. m economic agents labelled with the index j = 1 … m,
3. one good, "money," is a medium of exchange and unit of account.

At the start of the current time period, the economy's prevailing relative price vector $p = (p_1, \ldots, p_n)$, in terms of the unit of account. The economy's agents maximize their objective functions through exchange at these prevailing prices, subject to their transaction budget and goods transaction balance constraints. If the k^{th} good is money, p_k is the borrowing / lending interest rate for one unit of money for one time period. If the c^{th} good is a capital item, p_c is its rental rate for one unit for one time period. If the l^{th} good is labour, p_l is the wage rate per unit.

Formally, agents attempt to maximize their objective functions subject to constraints:

$\max f_j(x_1^j, \ldots, x_n^j)$, the j^{th} agent's objective function,

such that $\sum_{i=1}^{n} p_i x_i^j = 0$, j = 1 … m, the j^{th} agent's transaction budget constraint,

where $x_i^j > 0$ if the i^{th} good is demanded by agent j,

$x_i^j < 0$ if the i^{th} good is supplied by agent j,

$x_i^j = 0$ if the i^{th} good is not transacted by agent j,

and $\sum_{j=1}^{m} x_i^j = 0$, i = 1 … n, the i^{th} market's clearing condition

For intertemporal reasons, assume that at the start of each time period an agent is endowed with some goods, like labor (income) and money, that can be sold or used during the current period,
i.e. let:

$\bar{x}^j = (\bar{x}_1^j, \ldots, \bar{x}_n^j)$, be the j^{th} agent's endowment vector of goods.

Also, let the inventory of goods carried over from the previous time period be:

$ix^j = (ix_1^j, \ldots ix_n^j)$,

Namely for the j^{th} agent, it is the vector of non-perishable goods left over at the end of the previous time period, plus repaid money loaned plus interest and returned capital goods to the j^{th} agent.

If agent j is a producer of goods, let:

$px^j = (px_1^j, \ldots, px_n^j)$, be the current period's vector of production by the agent.

For non-producing agents, $px^j = (0, \ldots, 0)$

Then, the vector of goods available for agent j to sell during the current time period is:

$\underline{x}^j = (\underline{x}_i^j, \ldots \underline{x}_n^j) = \not{x}^j + ix^j + px^j$, i.e. endowments + inventories + production.

Finally, the agent's goods transaction balance constraints for the current time period are:

$\underline{x}_i^j + x_i^j \geq 0, i = 1 \ldots m.$

If agent j buys good i, x_i^j is a positive amount. If agent j sells good i, then x_i^j is a negative amount

Assume that relative prices did not adjust instantaneously to equate demand and supply in all markets at the end of the previous time period; that market clearing is a gradual process. Thus, agents might discover they are unable to buy and / or sell all they want (consistent with maximization of their objective functions subject to the constraints) in a given market at the prevailing relative prices during the current time period

The j^{th} agent can be considered to solve the following Static Recursive Algorithm while undertaking transactions. This procedure is a generalization of Robert Clower's Dual Decision Hypothesis set out in "The Keynesian Counter-Revolution: A Theoretical Appraisal."

<u>The Static Decision Algorithm for j^{th} Agent</u>.

1. Start with the price vector $p = (p_1, \ldots, p_n)$, and the agent's objective function, and the transaction budget constraint and goods transaction balance constraints in the constraint set.
2. Maximize the objective function subject to the relations in the constraint set to determine the desired quantities x_i^j, $i = 1 \ldots n$.
3. Is the agent constrained in any (further) market i?
    a. If yes, add the effective transaction constraint $x_i^j \leq \dot{x}_i^j$ or $x_i^j \geq \dot{x}_i^j$ to the constraint set depending on whether the agent is constrained as a buyer or seller. Thus, $\dot{x}_i^j$ is the amount the agent is actually able to transact on the i^{th} market. (It is negative if sold.) Go to step 2.
    b. If no, label the resulting quantities bought or sold as $\ddot{x}_i^j$, $i = 1\ldots n$.

The initial solution of the algorithm, when only the transaction budget and goods transaction balance constraints are in the constraint set, is denoted by the vector $x^j = (x_1^j, \ldots, x_n^j)$. It is called the notional demand / supply vector for the j^{th} agent.

The solution vector to the complete recursive algorithm, $\ddot{x}^j = (\ddot{x}_1^j, \ldots, \ddot{x}_n^j)$, is called the effective demand / supply vector of the j^{th} agent.
If the agent's notional demand or supply is constrained on a given market, the result on his effective demand in other markets will of course depend on his objective function. We list the following outcomes as being reasonable.

1. Consider the effect when the j^{th} agent can buy only $\ddot{x}_i^j < x_i^j$ on the i^{th} market.

If some of his demand "spills-over" into his demand for good h, we could get $\ddot{x}_h^j > x_h^j$.

Alternatively, if the agent decides to produce and sell less of a good s, being limited in the use of factor i, we get $\ddot{x}_s^j > x_s^j$.

2. Consider the effect when the j^{th} agent can sell only $\ddot{x}_i^j > x_i^j$ on the i^{th} market (a smaller negative number is greater than a larger negative number).

An agent who is a producer of good i might reduce purchases of good q, a factor in production; we could get $\ddot{x}_q^j < x_q^j$.

Alternatively, in a market r where the agent is also a buyer, we could get $\ddot{x}_r^j < x_r^j$, since to satisfy the transaction budget constraint, the agent will use less of some goods.

All agents participate in their Static Decision Algorithm during the current time period. At the end of the current time period, all markets are assumed to have cleared. The ability of agents to transact their desired amounts could depend on when they engage in the market during the current time period. The notional and effective amounts demanded and supplied by each agent in each market are net amounts at the prevailing relative prices.

Market clearing, total amount bought equals total amount sold, of effective demand / supply implies that in all markets i:

$$\ddot{x}_i = \sum_{j=1}^{m} \ddot{x}_i^j = 0, \; i = 1 \ldots n. \quad \text{so that} \qquad \ddot{x} = (\ddot{x}_1, \ldots, \ddot{x}_n) = (0, \ldots, 0).$$

<u>The Dynamic Adjustment Mechanism for Prices</u>.

Measure disequilibrium in the economy at the end of the current period by the differences between notional and effective demand / supply. Postulate that the prices prevailing in the next time period will reflect these differences.
In each market, divide agents into groups depending on whether they are buyers, sellers, or neither. Formally, for the i^{th} market let:

$J = (1 \ldots m)$, the integer index set of all agents.

$J_i^B = j \in J$ such that $\ddot{x}_j^j > 0$, the index set of agents buying good i,
$J_i^S = j \in J$ such that $\ddot{x}_j^j < 0$, the index set of agents selling good i,
$J_i^N = j \in J$ such that $\ddot{x}_j^j = 0$, the index set of agents not transacting good i.

$$J = J_i^B \cup J_i^S \cup J_i^N.$$

For the i^{th} market, the total amount of good i transacted (total amount bought = total amount sold) is:

$$\ddot{x}_i^T = \text{sum}(\ddot{x}_i^j) \text{ for } j \in J_i^B = -\text{sum}(\ddot{x}_i^j) \text{ for } j \in J_i^S \geq 0.$$

For a market i, consider agent $j \in J_i^B$, a buyer of good i. If $(x_i^j - \ddot{x}_i^j) > 0$, the agent's notional demand is greater than the agent's effective demand. If as a result of a "spill-over" effect $(x_i^j - \ddot{x}_i^j) < 0$, the opposite is true, and the agent's excess demand is effectively zero.

Define the effective excess demand for the i^{th} market by:

e_i^D = sum {max [(x_i^j - $\ddot{x}_i^j$), 0]} for agents j ∈ J_i^B.

If at price p_i, buyers in total want to buy more than their transacted amounts,
$e_i^D > 0$. Here, one might expect upward pressure on the future price for good i, with an increase in the total amount of good i transacted in the next time period.

Now consider agent j ∈ J_i^S, a seller of good i. If (x_i^j - $\ddot{x}_i^j$) < 0, the agent's notional supply is greater than the agent's effective supply, in absolute value. Conversely, with "spill-over" effects, (x_i^j - $\ddot{x}_i^j$) > 0, the opposite is true, and here the agent's excess supply is effectively zero.

Define the effective excess supply for the i[th] market by:

e_i^S = sum {min [(x_i^j - $\ddot{x}_i^j$), 0]} for j ∈ J_i^S.

If at price p_i, sellers in total want to sell more than their transacted amounts,
$e_i^S < 0$. Here, one might expect downward pressure on the future price for good i with a decrease in the total amount of good i transacted in the next time period.

The excess transaction vector is $e^T = (e_1^T, \ldots, e_n^T)$, with components $e_i^T = e_i^D + e_i^S$.

We are assuming that at the end of the current time period, buyers and sellers can negotiate or set the future price $\underline{p}_i$ that will prevail in the next time period. We posit a price adjustment mechanism for the i[th] market, in which transactions have occurred, of the form:

$$\underline{p}_i = p_i + \alpha_i (e_i^D / \ddot{x}_i^T) + \beta_i (e_i^S / \ddot{x}_i^T),$$

We specify two positive adjustment coefficients, one for effective excess demand and one for effective excess supply.

Since the excess transaction ratios, ($e_i^D / \ddot{x}_i^T$) and ($e_i^S / \ddot{x}_i^T$), are, respectively, non-negative and non-positive per definitions, $\alpha_i > 0$ and $\beta_i > 0$.

At the start of the current time period, we had a relative price vector
$p = (p_1, \ldots, p_n)$, that failed to clear all markets. At the start of the next period, we will have a relative price vector $p = \underline{p} = (\underline{p}_1, \ldots, \underline{p}_n)$, again with no guarantee that it will clear all markets.

My model is a framework for the household and firm sectors of a capitalist economy in which markets may or may not clear at the prevailing relative prices, $p = (p_1, \ldots, p_n)$. The number of goods (markets) is unspecified. For example, a mega-firm like Amazon sell over 12 million products out of inventories, and about 350 million products for affiliated firms. Similarly, the number of agents selling and buying goods (and services) is unspecified.

The labour market is differentiated by the skill set of each worker, no two of which are identical. Businesses can be distinguished by their size, location, debt and ownership structure, and product mix, even in oligopolistic industries. For information on the number of businesses consult the telephone directory's pre-internet Yellow Pages for any major city.

It is likely that the financial industry might be incorporated into the framework. What about the various government levels that receive a portion of business and household income as well as market transactions? In this case, a value must be assigned to the government's provision of goods and services. For example, how much does a mile of highway driving cost?

The dynamic adjustment mechanism for prices posited a price adjustment mechanism for the i^{th} market, in which transactions have occurred, of the form:

$$\underline{p}_i = p_i + \alpha_i \left(e_i^D / \ddot{x}_i^T\right) + \beta_i \left(e_i^S / \ddot{x}_i^T\right),$$

where $e_i^D >= 0$ is the excess demand for good, i, and $e_i^S <= 0$ is its excess supply, only one of which can be non-zero. The positive coefficients α_i and β_i determine the speed of price adjustment in the i^{th} market.

According to the well-known cobweb model, if a market's price fluctuates freely, its price adjustment process can be either convergent, divergent / oscillatory, or undamped oscillatory, depending on whether the demand schedule's absolute slope value is less, greater, or equal to the supply schedule's absolute slope value.

Numerous rivals produce similar but different goods in a monopolistically competitive market. A monopolistically competitive firm can fix its pricing within a narrow range because of its very elastic demand curve.

An oligopoly can occur when a few large companies control the majority of the market. Although these companies have considerable control on prices, one company's actions can have a direct impact on the others.

Investment cycles can arise from the aggregated market for capital and consumer products adjusting too quickly, as the Kaldor business cycle model showed. Could similar cycles occur depending on the value of the parameters α_i and β_i?

Clower, Barro and Grossman, and Grossman's Keynesian models, which I examined in my Economics 502 notes, combined assets, labor, and consumption into a single variable. Clower's dual-decision hypothesis explains Keynesian concepts like the consumption function, and investment an increasing function of changes in the level of output.

Section 5.3 An Evaluation of the Dual-Decision Hypothesis.

Nagatani states that these models of disequilibrium do "a better job of explaining Keynesian difficulties than its predecessors, which may be broadly described as the rigid-wage theory and the dynamic disequilibrium theory."

According to the rigid-wage hypothesis, employment levels adjust based on the money wage rate specified by labor contracts. According to the dynamic disequilibrium hypothesis, the notional equilibrium prices are unknown to market players. Therefore, it takes time to reach the full-employment equilibrium.

The amount of aggregate demand, however, cannot be explained by the dual-decision model. Instead, it "is solely concerned with the issues facing market coordination." Moreover, Clower and the others fail to provide a clear explanation of how the economic agents create the notional-demand functions.

My "Dynamic Adjustment Mechanism for Prices" and "Static Decision Algorithm" model the notional supply, effective supply, notional demand, and effective demand for every good by each agent and for the total economy. Various forms of labour, produced goods (durable and perishable), consumer and company inventories (stocks), capital goods, assets (claims on capital items), etc. are conceptually taken into consideration in my “Economy of ‘n’ Markets and ‘m’ Agents” with agents maximizing an objective function subject to a series of constraints.

According to Nagatani's interpretation of Keynes, what is required is an intertemporal macroeconomic model in which economic agents make mistakes on the quantity of durable products (stocks) they manufacture, hold, or buy. In other words, excess demand and supply for durables, both individual and collective, characterize long-lasting economic failures. “In order to handle this problem properly, one clearly needs a capital-theoretic formulation of individual behaviour (e.g. of a firm stating with some ‘wrong’ initial stocks and planning a sequence of corrective measures over time).”

As an operations research analyst for Canada Packers, I worked on inventory control. Plant-level inventory control was used for both production output, such as cans of chicken and ham, and production inputs, such as materials and supplies (chickens and hogs). The head office oversaw capital investments in new facilities and large machinery, and industrial process renovations.

I updated the plant managers on inventory management strategies, such as the Economic Order Quantity (EOQ) and Economic Production Quantity (EPQ), with my head office colleagues. The ideal inventory levels that minimize inventory costs (ordering plus holding) are determined using EOQ. EPQ considers the duration of manufacturing runs for optimizing production.

Higher interest rates resulted in lower desired inventories since the opportunity cost of maintaining inventories is largely influenced by them. Monetary policy by way of the Central Banks interest rate clearly influences the stocks of inputs and outputs held by a firm.

Historical sales trends of the items were smoothed out to remove any unpredictability or inconsistencies in order to forecast demand for the EOQ and POQ formulae. Based on the output items anticipated demands, the input and output inventories were linked. Linear programming maximized the value of the production mix given the constraints of available inputs.

Section 5.4 Was Keynes a Keynesian?

Throughout his General Theory, Keynes places a strong emphasis on entrepreneurial expectations and aggregate demand. In his final chapter, Nagatani notes, Keynes oddly clashes with this approach. Conventional macroeconomic theories are unable to address real-world issues due to their unrealistic assumptions. However, Keynes believes that conventional macroeconomic theories, with their Walrasian market clearing, flexible prices, will be effective when aggregate demand is adequate. This inconsistency “caused so much grief” to Keynes’ Keynesian followers.

Equating the real wage with the marginal output of labor was another flaw in Keynes' analysis. Nagatani states that “in the language of the dual-decision theorists, this equality means that firms never experience quantity constraints in the goods market.” By eliminating this unjustified equality, the dual-decision theorists were able to claim they had unified Keynes' theory.

The dual-decision theory describes the behavour of economic agents when markets are in disequilibrium during a depression. However, it does not address the presence or duration of depressions.

Keynes *General Theory* "should have described the state of equilibrium to which a sequence of stock adjustments was expected to converge."

I make a distinction between decisions under risk and those under uncertainty in my Economics 503 essay for Professor Nagatani, "Optimal Savings Under Risk and Uncertainty" (https://www.egwald.ca/macroeconomics/pdf/optimalsavings.pdf). With risk, the decision-maker is fully aware of the stochastic process's underlying random variable's distribution. As the sequential process progresses, the planner neither learns more about the distribution of the random variable, nor takes into account that more knowledge about this distribution will be known at later stages.

This distinction could be used to explain the variations in capital investment and inventory/production decisions. Even if it is updated as events happen, the data regarding the distribution of demand for an inventory item is comparatively well understood.

There is more uncertainty surrounding information regarding the viability and profitability of a capital investment. Decisions at the head office imply uncertainty, whereas decisions at the plant involve risk.

The catastrophe at the Fukushima Nuclear Power Plant, which was triggered by the Tōhoku earthquake and tsunami, highlights the crucial difference between risk and uncertainty when deciding whether to make a capital investment.

Chapter 6. Savings and Investment in Keynesian Temporary Equilibrium.

6.1 The Investment Function.

To paraphrase Nagatani, Keynes thought that when investing in capital goods, long-term projections and the condition of the asset markets were crucial factors. The consequences of financial regulations, shifts in liquidity preferences, and speculative activity are all apprehended by the asset market. Keynes' marginal efficiency of capital embodies long-term expectations of aggregate demand.

6.2 The Marginal Efficiency of Capital (MEC).

The projected rate of return on an extra unit of capital is known as the marginal efficiency of capital. In Keynes' words, it is "the rate of discount which makes the supply price of the capital asset equal to the present value of the prospective yield from it."

In other words, it's the internal rate of return (IRR) that equates the cost of acquiring a capital asset with the present value of its expected future earnings. It should not be confused with either the marginal product of capital, expected marginal return from an additional unit of capital, or the current rate of return on existing capital assets.

According to Keynes, shifts in MEC are a major factor in economic booms and collapses. Optimistic expectations correlate with a higher MEC causing economic expansion. Pessimistic expectations correlate with a lower MEC economic recession.

The MEC's volatility and related trade cycles are explained by the psychological elements that affect it.

6.3 A Simple Model of Investment Decisions.

Given the importance of the MEC to Keynes theory of dynamic aggregate demand, Nagatani derives a mathematical version from an optimal control, neoclassical model of investment.

In Chapter 2, Nagatani illustrated the classical dynamic optimization and its solution using Pontryagin Maximum Principle. I will state the problem and follow Chapter 14 of M. D. Intriligator, *Mathematical Optimization and Economic Theory*. This book was the text for my graduate mathematical economics course at Carleton University, 1975-77.

(1) Maximize $J = \int_0^T [pF(K,L) - wL - C(I)]e^{-rt}\,]dt$

Subject to:

(2) $dK/dt = I - \delta K$, $(\delta > 0)$, $K(0) = K_0 > 0$, $K(T) >= 0$, $I(t) >= 0$, $L(t) >= 0$.

A firm produces a commodity with the production function, F(K,L), monotonically increasing, continuous in both variables, and concave for (K, L) > (0,0). It operates in competitive markets, so the price of its commodity, p, and the wage paid workers, w, are fixed. It wishes to invest, I, in capital, K, and higher labour, L, to maximizes profits as represented by the function J.

The price of investment goods, q, depends on the quantity per unit of time it plans to install, q = q(I). Investment is a flow. The firm will incur costs setting up the capital items and increasing the rate investment; the cost to the firm is a convex function of I, C(I) = q(I) I. Assume p, w, and q are fixed. Note that K, L, and I are functions of time. The variables L and I are control variables, and K is the state variable.

Introduce the costate variable, λ(t), a shadow price. It's initial value, λ(0) represents the change in the optimal value of the objective functional due to a change in the initial value of the state variable, K_0.

The Maximum Principle involves defining the Hamiltonian function:

(3) $H(t; K, L, \lambda) = pF(K,L) - wL - C(I) + \lambda(I - \delta K)$

From the Maximum Principle and the Kuhn-Tucker conditions:

(4) $\partial H/\partial L = p F_L(k,L) - w <= 0$, $[p F_L(k,L) - w]\, L = 0$.

(5) $\partial H/\partial I = -C'(I) + \lambda <= 0$, $[-C'(I) + \lambda]\, I = 0$.

Solve (4) and (5) for the control variables, L and K as functions of w, p and λ:

(6) $L = L(w/p, K)$; $I = I(\lambda)$.

Then the optimal solutions of the state and costate variables as continuous functions of time and the initial value of the state variable:

(7) $K^* = K^*(t; K_0,r,p,w,)$, $\lambda^* = \lambda^*(t; K_0,r,p,w,)$,

where:

(8) $M = e^{-rt} H = e^{-rt}[pF(K,L) - wL - C(I) + \lambda(I - \delta K)]$,

(9) $d\mu/dt = - \partial M/ \partial K = -e^{-rt}[pF_K(K, L) - \lambda\delta]$,
(10) $\mu = \lambda\, e^{-rt} \rightarrow d\mu/dt = (d\lambda/dt)\, e^{-rt} - r\lambda\, e^{-rt}$,
(11) $d\lambda^*/dt = \lambda^*(r + \delta) - pF_K[K^*, L(w/p, K^*)]$,
(12) $dK^*/dt = \partial H/ \partial\lambda = I(\lambda^*) - \delta K^*$,
(13) $K^*(0; K_0, r, p, w) = K_0$,
(14) $e^{-rT} \lambda^*(T; K_0,r,p,w,)\, K^*(T; K_0,r,p,w,) = 0$,
(15) $\lambda^*(T; K_0,r,p,w,) >= 0$, $K^*(T; K_0,r,p,w,) >= 0$.

Let $J^*(K_0; r, w, p)$ be the maximum value of the discounted profits in (1). Then according to Nagatani, we know that:

(16) $\partial J^*(K_0; r,p,w) / \partial K_0 = \lambda^*(0; K_0, r,p,w,)$.

Therefore, $\lambda^*(0; K_0,r,p,w,)$, the optimal initial price, is the marginal worth of the initial capital stock, K_0. Thus, $\lambda^*(0; K_0,r,p,w,)$ is closely related of Keynes notion of MEC.

"Keynes MEC is defined as the rate of discount R for which the following equality holds:

(17) $C'(0) = \lambda^*(0; K_0,R,p,w,)$.

We can readily show that $\lambda^*(0; K_0,r,p,w,)$ is a monotonically decreasing function of r. Hence, if the interest rate r is less that R,

(18) $C'(0) = \lambda^*(0; K_0,R,p,w) < \lambda^*(0; K_0,r,p,w) = C(I^*)$,

with $I^* > 0$, whereas if R <= r, the inequality is reversed, and $I^* = 0$."

The "important fact is that the investment demand is determined by $\lambda^*(0; K_0,r,p,w)$, which measures the contribution of an additional unit of the initial capital stock to the profit sum.

Let's work out an example.

Production function: Cobb-Douglas: $Q = F(K, L) = K^{(1-\alpha)} L^{\alpha}$.

The price of investments goods: $q(I) = a\, I$. The cost function: $C(I) = q(I)\, I = a\, I^2$. The marginal cost of investment: $C'(I) = 2\, a\, I$.

Investment function $C'(I) + \lambda = 0 \rightarrow I(\lambda) = \lambda / (2\, a)$.

First order conditions (4) and (5)

(19) $p\, F_L\, (K, L) - w = p\, K^{(1-\alpha)}\, \alpha\, L^{(\alpha-1)} - w = 0$,
(20) $L^{(1-\alpha)} = (\alpha\, p/w)\, K^{(1-\alpha)} \rightarrow L = (\alpha\, p/ w)^{(1/(1-\alpha))}\, K$,
(21) $-C'(I) + \lambda = 2\, a\, I + \lambda = 0 \rightarrow I = \lambda / (2a)$.

From (12): (22) $dK/dt = -\delta\, K + I = -\delta\, K + \lambda / (2a)$

From (11): (23) $d\lambda/dt = - pF_K[K, L(w/p, K)] + \lambda\, (r + \delta) = - p\, (1-\alpha)\, K^{-\alpha}\, L^{\alpha} + \lambda\, (r + \delta)$,
(24) $d\lambda/dt = - p\, (1-\alpha)\, K^{-\alpha} = (\alpha\, p/ w)^{(\alpha/(1-\alpha))}\, K^{\alpha} + \lambda\, (r + \delta)$,

(25) $d\lambda/dt = - p (1 – \alpha) (\alpha p/ w)^{(\alpha / (1 - \alpha))} + \lambda (r + \delta)$,

(26) $d\lambda/dt = 0 K + \lambda (r + \delta) - p (1 – \alpha) (\alpha p/ w)^{(\alpha / (1 - \alpha))}$

Let's assign some values to the given variables.

Set: $\alpha = 0.85$, $r = 0.03$, $\delta = 0.05$, $w = 1$, $p = 1.3754$, $a = 2.5$, $1 / (2*a) = 0.2$, $Q = F(K, L) = K^{0.15}L^{0.85}$,

$C(I) = 2.5 I^2$, $C'(I) = 5I$, $I(\lambda) = \lambda / (2 a) = 0.2 \lambda$, $- p (1 – \alpha) (\alpha p/ w)^{(\alpha / (1 - \alpha))} = -0.5$.

The ordinary differential equations (ODEs) governing the dynamics of the model:

(27) $dK / dt = -0.05 K + 0.2 \lambda$,

(28) $d\lambda / dt = 0 K + 0.08 \lambda – 0.5$.

Boundary Conditions: 0 <= t < = 20; K(0) = 4 and λ(20) = 0.

The phase diagram plotter on my webpage at:
https://www.egwald.ca/linearalgebra/lineardifferentialequations.php#phasediagram indicates the homogenous system has a saddle point at the origin.

In matrix form write the systems as: d$\underline{x}$/dt = A$\underline{x}$ + $\underline{b}$, where $\underline{x}$ = (K, λ)', $\underline{b}$ = (0, -0.5)', and A is the matrix

(29) $$A = \begin{vmatrix} -0.05 & 0.2 \\ 0 & 0.08 \end{vmatrix}$$

The eigenvalues and eigenvectors:

(30) $\mu_1 = 0.08$, $\gamma_1 = (20, 13)'$; $\mu_2 = -0.05$, $\gamma_2 = (1, 0)'$.

The particular solution:

(31) $A x_p + \underline{b} = 0 \rightarrow x_p = (25, 6.25)'$.

The general solution:

(32) $K(t) = 20 c1 \exp(0.08 t) + c2 \exp(-0.05 t) + 25$.

(33) $\lambda(t) = 13 c1 \exp(0.08 t) + 6.25$.

Applying the boundary conditions $K(0) = K_0 = 4$, $\lambda(20) = 0$:

(34) c1 = −0.097104627, c2 = −19.05790746.

The remaining boundary values:

(35) K(20) = 8.3726, λ(0) = 4.9876.

Solution courtesy of AI Groc.

Time profiles of system variables.

In Figure 4, K vs I, net Investment = $I – \delta K > 0$ if $I > 0.4765$. The K trajectory has a maximum value of 9.5859.

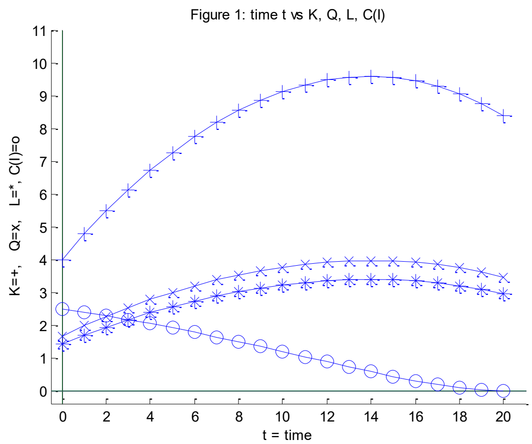


Figure 1: K, Q, L, C(I) as functions of time t.

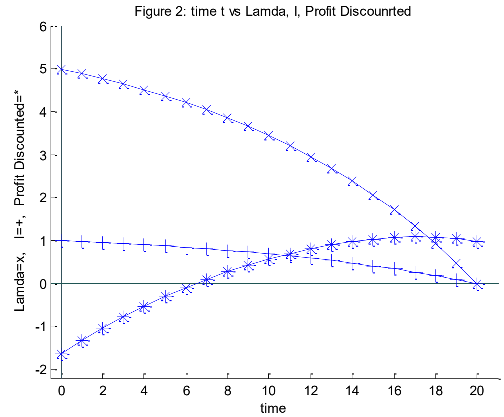


Figure 2: λ, I, Profits*exp(-rt) as functions of time

Phase diagrams:

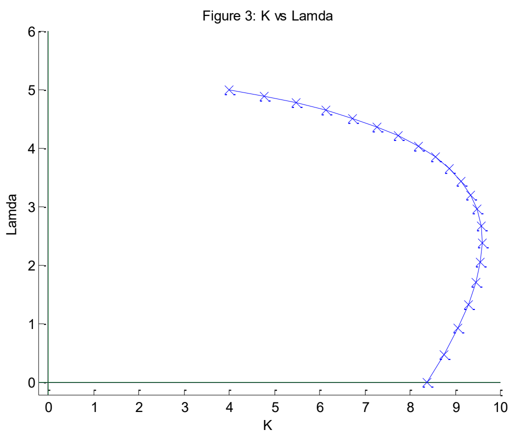


Figure 3: K vs λ.

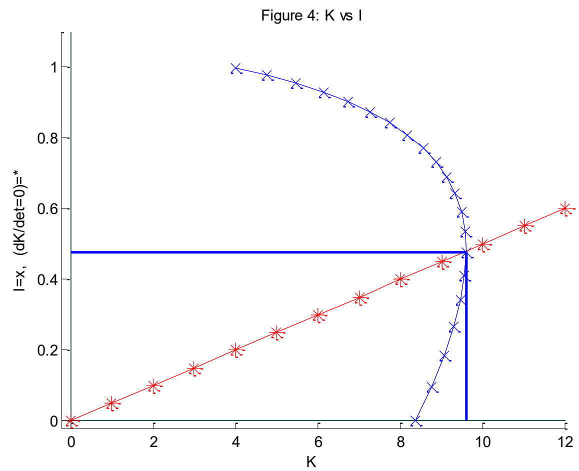


Figure 4: K vs I; red line δK intersects: (9.5859, 0.4765)

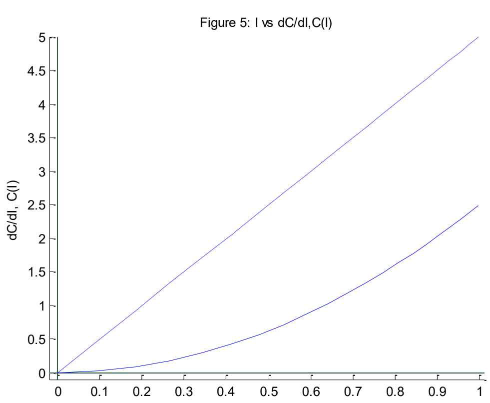


Figure 5: C(I), is a convex function of I.

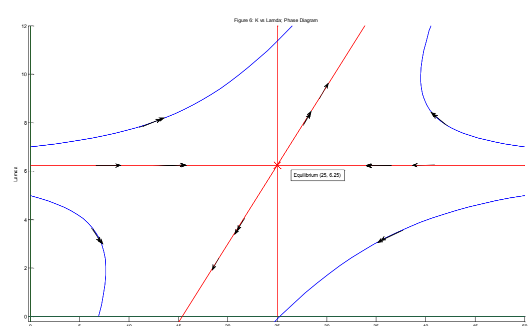


Figure 6: K vs λ Phase Diagram: Equilibrium (25, 6.25)

The model's K vs λ trajectory is in the lower left quadrant of the saddle-point equilibrium.

Nagatani states that Keynes' MEC is defined for the rate of discount R for which the following relation holds if $r < R$ and $I^*(t) > 0$:

(36) $$C'(0) = \lambda^*(0, K_0, R, p, w) < \lambda^*(0, K_0, r, p, w) = C'(I^*).$$

For K_0, R, p, w as defined above:

(37) $$C'(0) = 0 = \lambda^*(0, K_0, R, p, w) < \lambda^*(0, K_0, r, p, w) = 4.9881 = C'(I^*(0)) = C'(0.9976).$$

Total discounted profits (step function) ~= 5.3996 = Max J (wrt L and I) = $J^*(K_0, r, p, w)$.

Sensitivity analysis on initial value of K_0. Change only K(0) boundary condition: K(0) = 5, λ(20) = 0.

The general solution remains as:

(32) $$K(t) = 20\ c1 \exp(0.08\ t) + c2 \exp(-0.05\ t) + 25.$$

(33) $$\lambda(t) = 13\ c1 \exp(0.08\ t) + 6.25.$$

Applying the boundary conditions $K(0) = K_0 = 5$, $\lambda(20) = 0$:

(34) $$c1 = -0.097104627,\ c2 = -18.05790746.$$

The coefficient c1 is the same because λ(20) = 0 and the equation for λ(t) are unchanged.

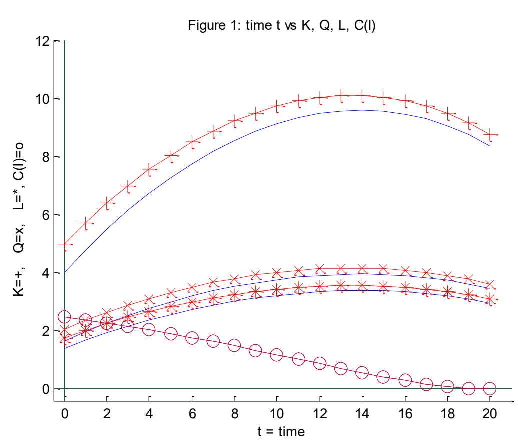
The remaining boundary values:

(35) $$K(20) = 8.7394,\ \lambda(0) = 4.9876.$$

Sensitivity Analysis: The increase in K_0 to 5 increases K(20) by 0.3679, and leaves λ(t) and λ(0) unchanged.

In the graphs below, the solid blue lines are the trajectories with $K_0 = 4$, red trajectories with $K_0 = 5$. No solid blue lines indicate the trajectories are the same.

Time profiles of system variables.



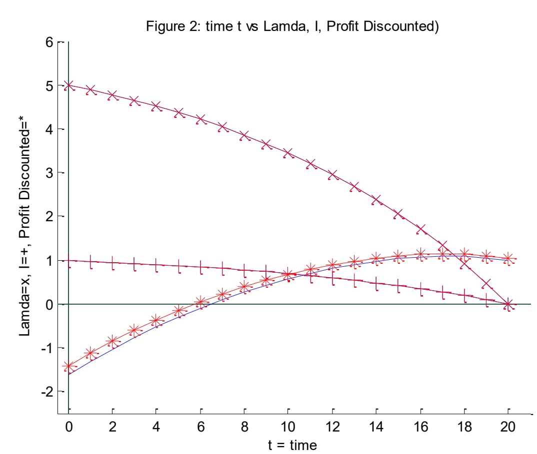
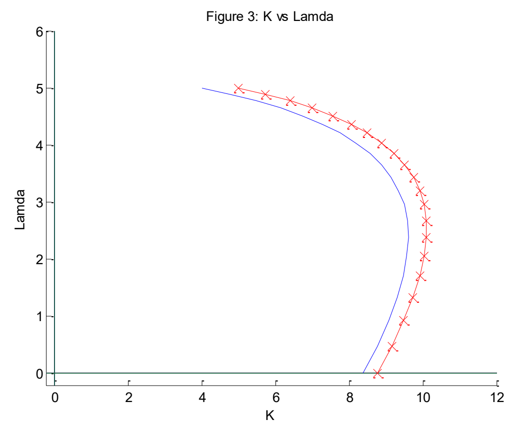


Figure 1: K, Q, L, C(I) as functions of time t.

Figure 2: λ, I, Profits*exp(-rt) as functions of time

Phase diagrams.



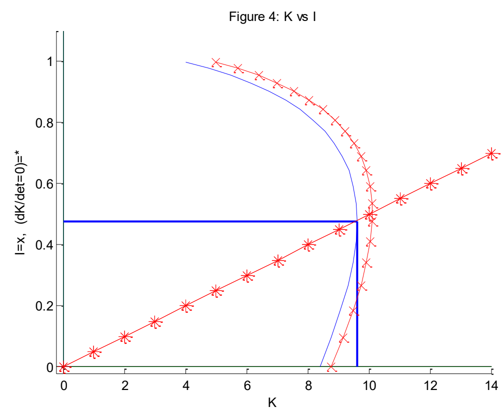
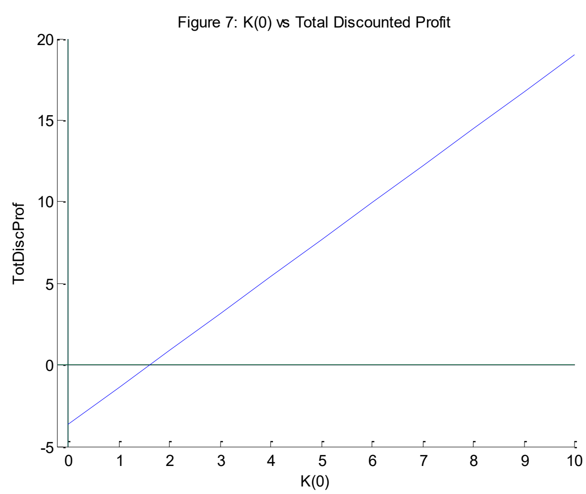


Figure 3: K vs λ.

Figure 4: K vs I; red line δK

Total discounted profits (step function) with K_0 = 5 ~= 7.6703= Max J (wrt L and I) = $J^*(K_0, r, p, w)$, an increase of 2.2707 from 5.3996 with K_0 = 4.

The results of this model contradict Professor Nagatani's conclusions on pages 97- 98.



Nagatani's figure 6.1 graphing K vs I differs from the above model's Figure 4. The assumption that λ(T) = 0 implies I(T) = 0. Over time, I(t) and λ(t) decrease, K(t) increases while net I(t) = I(t) – δ K(t) > 0, and decreases while net I(t) < 0. Nagatani's (K^*, I^*) when I(t) = δ K(t), is not an equilibrium point. It represents the time, t =~ 14, when output, Q, and the undiscounted value, p*Q, are at a maximum.

The values of the costate variable at t = 0, $\lambda^*(0, K_0, r, p, w)$, and the initial value of investment, $I^*(0)$, do not depend of K_0. The maximum value of the discounted profits, $J^*(K_0, r, p, w)$, is a linear function of K_0 with slope 2.2707 (Figure 7). Consequently, $J^*(K_0, r, p, w)$ does not have an optimal value with respect to K_0.

Nagatani's representation on pages 98 and 99 of the accelerator-type of investment theory make sense only if the point, (K^*, I^*) is a stable equilibrium node of the underlying dynamic structure.

6.4 Some Keynesian Modifications of the Investment Theory.

Nagatani examines Grossman and Arrow's efforts to incorporate Keynesian concepts into Section 6.3's neoclassical investment model. Estimating that it can sell all of its products at the price, p, the company solves its optimal control problem in accordance with Section 6.3 over a planning horizon of T years. After that, the capital stock will be obsolete.

(1) Maximize $J = \int_0^T [pF(k, L) - wL - C(I)]e^{-rt}\,]dt$

Subject to:

(2) $dK/dt = I - \delta K$, $(\delta > 0)$, $K(0) = K_0 > 0$, $K(T) >= 0$, $I(t) >= 0$, $L(t) >= 0$,

(3) $H = [pF(K, L) - wL - C(I) + \lambda(I - \delta K)]$,

(4) $\text{Profit} = p\,F(K, L) - wL - C(I)$,

where the variables, Profit, Q = F(k, L), K, L, and I are functions of time, 0 <= t <= T. At time t = 0, under the optimal control program, the firm sells $Q_0 = F(K_0, L_0)$ units at the price, p.

After some time when its output equals Q^*, at $t = t^*$, there is insufficient demand for its product. It can only sell Q^* units at the current price, p, for $t >= t^*$, investing to cover capital depreciation.

Use the example of Section 6.3 with a Cobb-Douglas production function: $Q = F(K, L) = K^{1-\alpha} L^{\alpha}$, $\alpha = 0.85$, $r = 0.03$, $\delta = 0.05$, $w = 1$, $p = 1.3754$, $a = 2.5$, $1 / (2*a) = 0.2$, $C(I) = 2.5\,I^2$, $C'(I) = 5I$, $I(\lambda) = \lambda / (2\,a) = 0.2\,\lambda$

With a planning horizon of T = 20, Figure 1 of Section 6.3 shows the optimal time profiles of K, L, Q = F(I, K), and C(I); Figure 2 the optimal time profiles of λ, I, and $M(t) = e^{-rt} H(t) = e^{-rt}$ Profits(t).

Evaluate the time profiles of K, L, I, and λ when $Q = Q^* = 3$ at time $t = t^* = 5$:

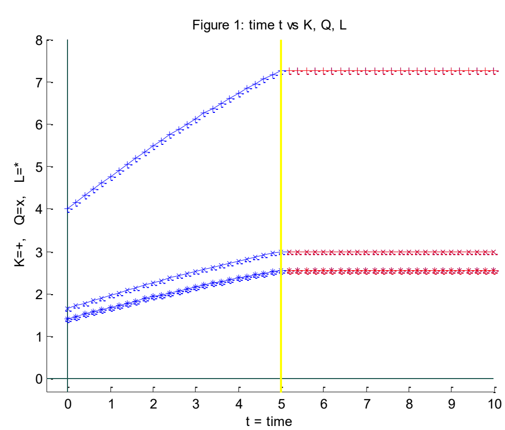
(5) $Q^* = 3$, $K^* = 7.27$, $L^* = 2.563$, $I^* = 0.8740$, $\lambda^* = 4.37$.

The firm's profit at $t^* = 5$ along the optimal trajectory: $\text{Profit}^* = -0.3513$.

The firm sells $Q^* = 3$ units when t > 5, but it invests only $I^\# = \delta K^* = 0.3631 < I^*$.

$\text{Profit}^\# = p\,Q^\# - w\,L^* - C(I^\#) = 1.3754 * 3 - 1 * 2.563 - 2.5 * 0.3631 * 0.3631 = 1.2332 > \text{Profit}^*$.

Time profiles of the system variables.



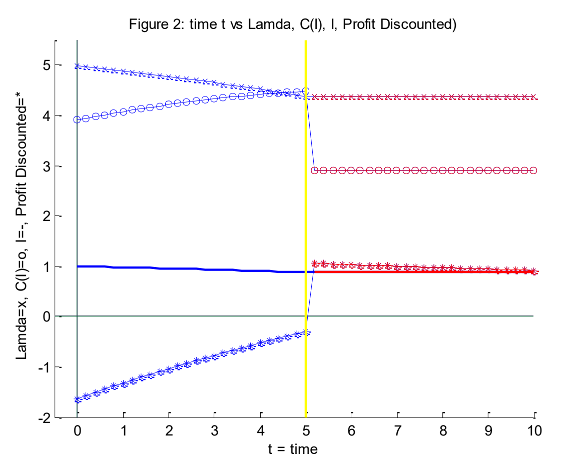


Figure 1: K, Q, L as functions of time t.

Figure 2: λ, C(i), I, Profits*exp(-rt) as functions of time

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