Optimal Savings Under Risk and Uncertainty

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I. Introduction.

To set the context for the economic problem of optimal savings under risk and uncertainty, consider a hypothetical economy with a planner who must decide, in each time period what portion of economic output is to be consumed and what portion is to be retained as capital. The planner seeks to make these allocations so as to maximize the utility (in the sense of satisfaction) of consumption over some planning horizon. Because of uncertainties due to nature and errors of observation, the planner does not know how much output will be available in the next time period by way of production with a given amount of capital during the current time period.

The planner will formulate a strategy to make optimal allocations of output over time given the planner's utility as a function of consumption over the current and future time periods. Uncertainty about the evolution of the economy necessitates taking into consideration at the outset this indeterminate knowledge when the optimal plan is calculated.

If the planner has the ability to learn more about the economy as it unfolds, i.e. gain more knowledge about the distribution of the randomness of the production process, the intertemporal consumption plan can be modified in each subsequent time period if necessary.

Within these parameters, we will look for answers to following questions.

1. Does an optimal consumption plan exist? How does it depend on the degree of uncertainty and the length of the planning horizon?
2. Does an optimal consumption plan result in a long run equilibrium for the distribution of capital?
3. Is the capital path corresponding to the optimal consumption path stable?
4. What effect does expectation formation and the decision criterion have on the optimal consumption plan?
5. Does an optimal expectation formation and adjustment transformation exist?
6. Do expectations reach some form of steady state (asymptotic) with respect to the adjustment transformation?

Questions 1. – 3. Are answered by way of reference to the literature. Questions 4. And 5. Are partially answered by analyzing a simple model.

II. Review of Dynamic Programming.

Using the methods of Richard Bellman and Robert Kalab in Dynamic Programming and Modern Control Theory, we will restate the problem outlined in Section 1. in the form of an adaptive control theory problem. Before we do this, we review some of the basic definitions and concepts of dynamic programming.

1. Dynamic System.
   A dynamic system consists of a state vector X_n and a rule for determining its value at any time n.

2. Transformation.
   A transformation T is a rule for determining the value of the state vector at time n+1 given its value at time n. We write:
   $$X_{n+1} = T(X_n).$$

3. Multistage Process.

   A multistage process is a sequence of state vectors:
   $$[X, X_1, X_2, \ldots, X_n, \ldots] \text{ where } X_{n+1} = T(X_n).$$

4. Multistage Decision Process.
   Suppose the transformation T depends not only on X_n, but also on a vector at time n, C_n, which the planner can control. Then C_n is called a decision variable, and: $X_{n+1} = T(X_n, C_n)$, $0 \le C_n \le X_n$.

5. Return Function.
   Suppose the planner wants to choose the value of the decision variable C_n so as to maximize some function of the state and decision variables:
   $$U(X, X_1, \ldots; C, C_1, \ldots).$$
   The function U is called the return (utility) function. We will consider return functions for the planner of the form:
   $$U_N = \sum_{n=0}^{N} U(X_n, C_n), \; 0 \le C_n \le X_n.$$

6. Policy.
   A policy is a function that relates the decision variable C_n to the state variable X_n. An optimal policy maximizes that return function.

7. Principle of Optimality.
An optimal policy has the property that whatever the initial state and decisions made so far, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Define $f_N(X)$ as the total maximum N-stage return obtained starting in state X using an optimal policy. Then the recursive principle of optimality is:

$$f_N(X) = \max \{ [U(X, C_0) + f_{N-1}(T(X, C_0))] \text{ w.r.t. } C_0\},$$
$$\text{with } 0 \leq C_0 \leq X_0,\ X_1 = T(X_0, C_0).$$

8. Stochastic Multistage Decision Algorithm.
To allow for stochastic (random) effects, consider transformations of the form:

$$T = T(X, C, r), \text{ i.e. } X_{n+1} = T(X_n, C_n, r_n),\ \ 0 \leq C_n \leq X_n,$$

with the r_n independent random variables having a common distribution ν.

R. Duncan Luce and Howard Raiffa in Games and Decisions have shown that in decision making under risk, expected return is the correct measurement of return for comparing alternate decision policies.

Thus, if $f_N(X)$ is the total expected N-stage return using an optimal policy, the principle of optimality is:

$$f_N(X) = \max \{[U(X, C_0) + \int f_{N-1}(T(X, C_0, r_0))\, d\nu] \text{ w.r.t. } C_0\},$$
$$\text{with } 0 \leq C_0 \leq X_0,\ X_1 = T(X_0, C_0).$$

9. Adaptive Control Process.
In the stochastic decision process, the planner had knowledge of the distribution of the random variable r. Under adaptive optimization, the planner has some prior estimate of the distribution of the random variable. As the sequential process unfolds and information becomes available, the planner can revise estimates of the random variable based on these observations.

Let X be the state variable, C the decision variable and r the random variable with a fixed but unknow distribution. If ν is the planner's *a priori* estimate of the fixed but unknown distribution of r, the state of the adaptive control process is (X, ν). As before, the planner wants to maximize the expected value of the return function with the addition of the random variable:

$$U_N = \sum_{n=0}^{N} U(X_n, C_n, r_n), \; 0 \le C_n \le X_n.$$

However, now we have two transformations after a choice of C_0,

$X_1 = (X, C_0, r_0)$, and

$\nu_1 = S(r; X, X_0, r_0, \nu)$.

If $f_N(X, \nu)$ is the expected value of U_N obtained by using an optimal policy starting with the state (X, ν) over N stages, the principle of optimality is:

$$f_N(X, \nu) = \max \{[U(X, C_0) + \int f_{N-1}((T(X, C_0, r_0), \nu_1)) \, d\nu_1] \text{ w.r.t. } C_0\}$$
$$\text{with } 0 \le C_0 \le X_0, X_1 = T(X_0, C_0).$$

It is of interest to know under what conditions the planner's optimal policy of the adaptive control process will converge to the optimal policy of the equivalent stochastic control process of definition 8. where the distribution of the random variable is known.

III. Review of the Literature.

The models in the economic literature to-date (1973), Emund Phelps, Jerusalem Levhari and T. N. Srinivasan, William Brock and Leonard Mirman, and Leonard Mirman, which attempt to generalize optimal growth models to uncertainty, are all examples of stochastic decision processes.

Although these authors claim to be analyzing models with uncertainty, Luce and Raiffa in Games and Decisions define their models as decision making under risk. That is, the planner in their models has complete knowledge of the distribution of the random variable under-lying the stochastic process.

The planner neither learns more about the distribution of the random variable as the sequential process unfolds, nor takes into consideration when making decisions that more knowledge about this distribution will be known at later stages.

Edmond Phelps in "the Accumulation of Risky Capital: A Sequential Utility Analysis" was the first author to use the theory of dynamic programming to investigate optimal consumption decisions in a discrete time stochastic growth model. At each stage, n, in Phelps' model, the planner (an individual) chooses to consume some amount C_n of available capital X_n. The rate of capital growth (capital gains) is a random variable r_n, with known, independent distribution at each stage, i.e. the known distribution of r_k is independent of the equivalent distribution of r_l for any pair of time periods k and l over the planning horizon.

Also, the individual receives an amount of non-wealth income at the end of each time period, Thus, the amount of capital available in the succeeding time period satisfies the difference equation:

$$X_{n+1} = r_n (X_n - C_n) + y.$$

The individual's utility function has the form:

$$U_N = \sum_{n=0}^{N} U(C_n)\, \delta^n,\ 0 \le C_n \le X_n,\ 0 < \delta < 1.$$

Although this form of the utility function has been criticized because of its properties, intertemporal independence and positive "time preference" rate δ, its use is prevalent in optimal growth models (since the time of Ramsey) because of analytical convenience. The objective of the planner is to maximize the expected value of U_N subject to the difference equation and the constraints.

Using dynamic programming, Phelps finds that optimal consumption is an increasing function of age (planning horizon) and capital. When Phelps restricts himself to certain monomial utility functions, he finds that:

> Consumption cannot be expressed as a function of aggregate income because expected wage income and expected capital income have different variance, whence different impact upon the level of consumption.

In "Optimal Savings Under Uncertainty", Jerusalem Levhari and T. Srinivasan extend the analysis of Phelps' model with he simplifying assumption that the planning horizon is infinite and the wage income is zero. Their main result is to derive a set of sufficient conditions for a consumption policy to be optimal. With capital stock X_n a random variable, the consumption policy, at any stage n, is a function of existing capital X_n, i.e. $C_n = g(X_n)$.

A feasible consumption policy satisfies the non-negative constraints $0 \le C_n \le X_n$. It is said to be optimal if for any other feasible consumption l, g overtakes l. That is if $C_n = g(X_n)$ and $\underline{C}_n = l(X_n)$ are feasible consumption policies then:

$$E\left[\ \sum_{n=0}^{N}(U(C_n) - U(\underline{C}_n))\, \delta^n\right] > 0,$$

for all time T > some time T_0, where E denotes the expected value.

The authors prove that if a feasible consumption policy g satisfies the conditions:

1. $U' [g(X_n)] = \delta\, E[r\, U' (g((X_n - g(X_n)\, r))\,]$, for all n,
2. $E [\, \delta^n\, U'(g(X_n))\, X_n\,] \rightarrow 0$ as $n \rightarrow \infty$,

then g is optimal policy. Equation 1., which is also a necessary condition of an optimal consumption policy, says that:

> the marginal utility of current consumption equals the expected value of the product of the discounted marginal utility of next period's consumption and the gross rate of return r to saving.

The second (transversality) condition seems to be necessary to prevent:

1. The amount of capital becoming infinitely large and "overshadowing" the decrease in the discount factor and the marginal utility of consumption, and
2. The marginal utility of consumption from becoming infinitely large as a result of the amount of capital becoming very small, and thus "overshadowing the decrease in capital and/or the discount rate. This would seem to be satisfied for a reasonable utility function, if the distribution of the capital random variable X_n reaches some asymptotic steady state distribution which is bounded, in a probabilistic sense, away from zero and infinity. This condition seems to anticipate the results of Brock and Mirman.

William Brock and Leonard Mirman in "Optimal Economic Growth and Uncertainty: The Discounted Case" investigated the asymptotic properties of one-sector optimal stochastic growth models. They motivate their analysis with reference to the well-known knife-edge property of the optimal consumption policy of the deterministic model. If the Euler conditions are followed after a perturbation, the result is instability and eventual annihilation. Brock and Mirman show that if the planner takes the possibility of perturbations into consideration at the outset, the stability properties of the optimal consumption policy are preserved.

To illustrate the scope of their model, we will write it down in some detail.

Model:

$$\max \quad E\, [\, \textstyle\sum_{n=0}^{N} U(C_n)\, \delta^n],\;\; 0 < \delta < 1,\;\; 0 \le C_n \le X_n$$

$$C_n + X_n = f(X_{n-1}, r_{n-1}),\;\; n = 1, 2, \ldots$$

Properties of the Utility function U:

$$U'(C) > 0,\ U''(C) < 0,\ U'(0) = +\infty,\ \ C > 0.$$

Properties of the Random Variable ν:

At each stage n, the r_n are independent, with common distribution ν, taking values in the real line. The support of ν is contained in the interval [α, β], where $0 < \alpha < \beta < \infty$. Also, for all $X > 0$, $\infty > f(X, \beta) > f(X, \alpha) > 0$.

Properties of the Production Function f:

$$f(0, r) = 0,\ f_1(X, r) > 0 \text{ and } f_{11}(X, r) < 0 \text{ for } 0 < X < \infty,$$

$$f_1(0, r) = \infty,\ f_1(\infty, r) = 0,\ f \text{ and } f_1 \text{ are continuous in both arguments.}$$

These conditions are sufficient to show there exists a unique solution to the maximum problem.

Denote the optimal consumption policy by:

$$C_n = g(f(X_{n-1}. r_{n-1})),$$

and the optimal capital accumulation policy by:

$$X_n = f(X_{n-1}, r_{n-1}) - g(f(X_{n-1}, r_{n-1})),$$

$$= h(f(X_{n-1}, r_{n-1})) = H(X_{n-1}, r_{n-1}).$$

It is easy to show that the optimal policies, g and H, respectively for consumption and capital, are continuous, increasing functions of the capital stock.

From Levhari and Srinivasan, we get the optimal policy's necessary condition:

$$U'[g(f(X_{n-1}, r_{n-1}))] = \delta\, E[U'(g(f(X_n, r_n))\, f_1(X_n, r_n))].$$

Associated with each random variable X_n of the stochastic process $(X_0, X_1, \ldots)$ is a probability measure μ_n defined by:

$$\mu_n(B) = P_r\{X_n \in B\}.$$

That is, $\mu_n(B)$ is the probability that the optimal capital belongs to the set B, given the initial capital stock X_0. Then, if the probability transition function P(X, B) is defined as the probability that the capital stock is in the set B one stage after it started in state X, a steady-state measure μ must satisfy:

$$\mu(B) = \int P(X, B)\, \mu(dX).$$

For a given measure μ_n (associated with X_n), define the distribution function $F_n(X)$:

$$F_n(X) = P_r\{X_n < X\} = \mu_n([0, X)).$$

Brock and Mirman show that there exists a distribution function F(X) such that for arbitrary initial stock X_0, the distribution functions $F_n(X)$ converge uniformly to F(X). Moreover, in a separate paper, Mirman has shown that the measure μ_{inf} associated with F(X) is unique, satisfies the stead-state property, and that the probability of the capital stock becoming zero or infinitely large is zero.

IV. Extensions Under Uncertainty.

In order to gain some insight into how the choice of a decision criterion and expectation formation affect the stochastic growth model, we will analyze a very simple model considered by Levhari and Srinivasan. Their chosen utility function is the constant elasticity function with the form:

$$U(C) = (1/(1 - \alpha))\, C^{(1 - \alpha)}, \text{ with } \alpha > 0.$$

The objective is to:

$$\max \quad E\left[\sum_{n=0}^{\inf} U(C_n)\, \delta^n\right], \; 0 < \delta < 1, \; 0 \le C_n \le X_n, \text{ where}$$

$$X_{n+1} = (X_n - C_n)\, r_n.$$

Levhari and Srinivasan have shown there exists an optimal consumption policy of the form:

$$g(X) = \lambda X, \text{ where } \lambda \text{ satisfies}$$

$$(*) \qquad (1 - \lambda)^{\alpha} = \delta\, E(r^{(1 - \alpha)}).$$

In our model of optimal savings under uncertainty, to maintain $0 \le \lambda \le 1$, we assume that the feasibility condition:

$$\delta\, E(r^{(1 - \alpha)}) < 1,$$

is satisfied in the following analysis. Also, the planner assumes that the random variable r, the states of nature, can take only the values 1/2 or 3/2. However, the true distribution of r is unknown to the planner. That is, the model involves decision making under uncertainty.

The question then arises—what is the proper decision criterion to use? Luce and Raiffa consider a decision criterion as

> well defined if and only if it prescribes a precise algorithm which, for any decision problem under uncertainty, unambiguously selects the act (policy) which is tautologically termed optimal according to the criterion.

Four decision criteria are considered here.

<u>The Pessimist's Criterion</u>:

The pessimist planner will try to protect against the "worst" that could occur. In our model, this is the random variable r always taking the value 1/2. From (*) we see that setting the optimal policy will set:

$$\lambda_p = 1 - [\delta\ (1/2)^{(1-\alpha)}]^{1/\alpha}.$$

<u>The Optimist's Criterion</u>:

The optimist planner assumes the "best" will occur. In our model, this is the random variable always taking the value 3/2. The optimal policy will set:

$$\lambda_o = 1 - [\delta\ (3/2)^{(1-\alpha)}]^{1/\alpha}.$$

<u>The "Normal" Criterion</u>:

Here, the planner assumes the two possibilities are distributed symmetrically (e.g. normal or uniform distribution), and uses the mean value. Thus:

$$\lambda_N = 1 - \delta^{1/\alpha}.$$

<u>The "Principle of Insufficient Reason" Criterion</u>:

If completely ignorant as to which state might obtain, the planner should behave as if the states are equally likely to occur:

$$\lambda_I = 1 - [\delta\ (1/2)((1/2)^{(1-\alpha)} + (3/2)^{(1-\alpha)})]^{1/\alpha}.$$

We see that the optimal consumption policy of the planner depends on the decision criterion used. When we compare the policies, we get:

$$\lambda_P \left\{\substack{\leq \\ = \\ >}\right\} \lambda_N \left\{\substack{\leq \\ = \\ >}\right\} \lambda_o \quad \text{as} \quad \alpha \left\{\substack{\geq \\ = \\ <}\right\} 1 \quad \text{and}$$

$$\lambda_I \left\{\substack{\leq \\ = \\ >}\right\} \lambda_N \quad \text{as} \quad \alpha \left\{\substack{\geq \\ = \\ <}\right\} 1.$$

These four decision criteria seem simplistic because the planner usually has some vague information about the true states of nature, information which the criteria do not use.

Luce and Raiffa examine axioms which reasonable decision criteria should fulfill. They show that to each criterion which resolves all decision problems under uncertainty in such a manner as to satisfy these axioms:

> there is an appropriate *a priori* distribution over the states of nature [independent of the acts (policies) in the problem] …, such that an act is optimal (according to the criterion) only if it is best against this *a priori* distribution.

Therefore, the planner's objective is to generate this *a priori* distribution from the information available about the state of nature starting at the plan's inception. In The Foundations of Statistics, Leonard Savage has shown that using hypothetical experiments, one can transform vague information concerning the states of nature into an explicit *a priori* distribution.

The strategy to be used by the planner in our simple model is now clear; at each stage, generate a probability distribution over the states of nature and then calculate the optimal consumption policy.

Suppose that after due deliberation, the planner decides the probability of the random variable taking the value 1/2 is γ, and the probability of 3/2 is necessarily $(1 - \gamma)$.

With λ defined as in equation (*) above, the optimal consumption policy then sets:

$$\lambda = 1 - [\delta\,(\gamma\,(1/2)^{(1-\alpha)} + (1-\gamma)\,(3/2)^{(1-\alpha)})]^{1/\alpha}.$$

An immediate conclusion is that the optimal consumption policy varies continuously with the subjective distribution, the variable γ. Thus, if the planner's information and intelligence is sufficient (e.g. maximum likelihood estimation?) for the subjective distribution over the states of nature to converge to the true distribution (assuming it exists), the optimal consumption policies will converge to the true optimal consumption policy.

The possibility that the planner will eventually (asymptotically) reach the true optimal consumption policy supports the robustness of the optimal planning models under risk.

If we solve the difference equation:

$$X_n = (X_{n-1} - \lambda X_{n-1})\, r_{n-1},$$

we get $$X_n = (1 - \lambda)^n X_0 \prod_{t=0}^{n-1} r_t,$$

and $$E(X_n) = (1 - \lambda)^n X_0 [E(r)]^n.$$

If a different consumption policy is used at each stage, we get:

$$\underline{X}_n = [\, \prod_{t=0}^{n-1}(1 - \lambda_t)\,]\, X_0\, [\, \prod_{t=0}^{n-1} r_t\,],$$

and $$E(\underline{X}_n) = E\, [\prod_{t=0}^{n-1}(1 - \lambda_t)]\, X_0\, [E(r)]^n.$$

Generally, at any stage, the expected quantity of capital in the uncertainty model will not equal the expected quantity of capital in the equivalent risk model. A sufficient condition for the expected values to be equal is that the λ_t be independently distributed at each stage with common mean λ. This would require that the planner's subjective distribution over the states of nature be independently distributed at each stage with an expected value (in the space of distributions) equal to the true distribution.

Brock and Mirman have shown that the distributions of capital stock X_n converge to a steady-state distribution. A sufficient condition for:

$$\lim_n E(\underline{X}_n) = \lim_n E(X_n)$$

is that the distributions of capital stock $\underline{X}_n$ converge to the same steady state distribution. However, we have not been able to prove this result here.

A supplementary extension would be to derive similar results for the general model of Brock and Mirman. As we have already indicted, a necessary condition for a consumption policy to be optimal is that:

$$U'[g(x\,)] = \delta \int U'[g\,(f(X\text{-}g(X, r)\,)]\, f_1(X - g(X), r)dv(r).$$

This equation can be written in its implicit form as:

$$G(X, g, v) = 0,$$

where X takes values in the real line, g takes values in the Banach space of bounded function on the real line, and v takes values in the Banach space of probability measures defined on the Borel subsets of the real line.

It is easy to show that the function Z = G(., ., .) is continuous in all its arguments. A theorem from Functional Analysis (e.g. Lazar Lusternik and V. J. Sobolev in Elements of Functional Analysis) proves that there exists a continuous function T such that for fixed X, g = T(ν) is equivalent to G(X, g, ν) = 0.

That is, the optimal consumption policy varies continuously (in the space of consumption policies) with respect to the probability measure ν. In other words, if the planner's subjective distribution converges to the true distribution, the optimal consumption will converge to the true optimal consumption policy.

Here again, we were unable to derive results about the time profiles or asymptotic properties of the capital stock.

V. Conclusions.

Preliminary results in this paper suggest that "groping" for the "true" optimal plan is a valid strategy for our models' planners to employ. However, expectations about the states of nature were formed in a subjective manner. We were unable to obtain a reasonable transformation of expectations from one stage to the next based solely upon variables observed in the single time period.

It was found that if the expectations converge to the true distribution over the states of nature, the corresponding optimal consumption policies converge to the true optimal policy.

However, we were unable to derive similar properties for the distributions of capital stock. We believe that it would be possible to show, with more analysis, that the expected values of capital stock converge to the expected value of the limiting distribution of capital stock corresponding to the equivalent model under risk.

This paper has focused on uncertainty in the production function. Further extensions would be to consider uncertainty in the utility function and, particularly, the costs and benefits of obtaining information.

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